Nonlinear σ model approach for level correlations in chiral disordered systems

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(Received 10 March 2004; revised manuscript received 11 October 2004; published 30 December 2004)

We study level correlations of disordered systems with chiral unitary symmetry (AIII symmetry). We use a random matrix model with a finite correlation length to derive a supersymmetric nonlinear σ model. The result is compared with existing results based on other models. Using the methods of Kravtsov and Mirlin **(**Pis'ma Zh. Fksp. Teor. Fiz. **60**, 645 (1994) [Sov. Phys. JETP **60**, 656 (1994)]**)** and Andreev and Altshuler [Phys. Rev. Lett. **75**, 902 (1995)], we calculate the density of states and two-level correlation function. The result is expressed using the spectral determinant as in traditional nonchiral systems. We discuss the renormalization of the mean level spacing which is not present in the traditional systems.

DOI: 10.1103/PhysRevE.70.066147 PACS number(s): 05.30.−d, 05.45.−a, 72.15.Rn, 71.30.+h

I. INTRODUCTION

The classification of disordered systems is based on symmetries of the Hamiltonian. According to invariance properties under time-reversal and spin rotation, three symmetry classes—unitary, orthogonal, and symplectic—are well known since the work by Wigner and Dyson [1]. The modern classification is based on the notion of symmetric spaces [2] and indicates that ten universality classes exist. Although there was an early effort at a universality classification in the 1980s [3], the additional seven classes did not attract much attention until physical applications were found [4,5]. The importance of chiral symmetry in disordered systems was first noticed in Ref. [4] by using random matrix theory (RMT) in the context of quantum chromodynamics (QCD) and mesoscopic quantum wires. In systems with chiral symmetry, eigenvalues of the Hamiltonian appear in pairs $\pm\epsilon$ and the origin $\epsilon=0$ plays a special role for level correlations.

In order to analyze such systems, the supersymmetry method [6] is known to be a useful tool for both perturbative and nonperturbative calculations. This method allows one to obtain a nonlinear σ model with supermatrix fields as effective modes. One can discuss weak localization effects using perturbation theory, where an expansion in terms of diffusion propagators (29) is performed. A diagrammatical interpretation is thus possible, and weak localization implies a large conductance $g\gg 1$, where g is proportional to the diffusion constant in the propagator. The localization property can also be discussed using the renormalization group method. This expansion is justified only for nonzero modes $q\neq 0$ in the propagator. The zero-mode sector contains a totally different contribution and gives the ergodic result $g=\infty$. Using the zero mode, we can calculate level correlation functions scaled in terms of the mean level spacing [6]. The result is nonperturbative, parameter-free, and universal. We know that treating the zero mode perturbatively gives only the asymptotic form of the exact result.

Thus it is important to notice the different roles of the zero and nonzero modes. At finite g, the nonzero modes modify the universal result of level correlation functions [7–9]. Kravtsov and Mirlin (KM) treated the zero and nonzero modes separately and found finite-g corrections to the universal result [8]. Due to technical problems, the result was restricted to the domain $z\ll g$ where z is the scaled energy variable. Using another method, Andreev and Altshuler (AA) considered the domain $z\gg 1$ where the perturbative expansion makes sense [9]. They reached the nonperturbative regime by noticing the existence of a set of nontrivial saddle points. Considering the expansion around two saddle points the result was expressed using the determinant of the diffusion propagator, which is called the spectral determinant in the literature. Although their method did not treat the zero and nonzero modes separately, it was shown in Ref. [10] that the separation, just as in KM's method, gives the same result.

Using the derived result, the authors in Ref. [9] found a smearing of the singularity at the Heisenberg time in the form factor (the Fourier transform of the two-level correlation function). Furthermore, the use of the spectral determinant represents a link from disordered to chaotic systems. The authors in Ref. [11] noticed that a similar treatment can be applied to general chaotic systems just by replacing the diffusion operator in the spectral determinant by the Perron-Frobenius operator. For a chaotic system, the expression of the determinant using the trace formula was discussed in Ref. [12]. Thus the expression using the spectral determinant is important for a unified treatment of disordered and chaotic systems. The result was applied to critical statistics [13] and the relation to the density-density correlation in the Calogero-Sutherland model at finite temperature was discussed.

In this paper we consider systems with chiral unitary symmetry. Starting from a chiral random matrix model with a finite correlation length, we derive a nonlinear σ model and calculate the density of states (DOS) and two-level correlation function (TLCF). Our aim in this paper is not to discuss a specific model but to discuss the generic properties of chiral symmetric systems. Actually the σ model we use in this paper is believed to be applicable to a broad range of physical systems and we discuss the relation to other σ models for specific systems. Then, we calculate the DOS and TLCF using a nonperturbative method which is equivalent to both methods of KM and AA. Our method is similar to that in

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Ref. [10] and the zero and nonzero modes are separated explicitly. For chiral symmetric systems, the calculation using the KM method has been carried out in Ref. [14]. In contrast to the approach in Refs. [8,14], we integrate the zero mode first, and then treat the nonzero modes perturbatively. The advantage of this method is that all domains are treated in a unified way. We also discuss the effect of the DOS renormalization, which is specific for nonstandard symmetry systems. We restrict our discussion to chiral unitary symmetry (AIII symmetry) and the extentions to other chiral symmetric classes, chiral orthogonal (BDI) and chiral symplectic (CII), will be discussed elsewhere.

The organization of this paper is as follows. In Sec. II, starting from the random Hamiltonian, we derive the supersymmetric nonlinear σ model. It differs from the traditional σ model written in terms of a supermatrix Q by symmetries of the matrix and the presence of an additional term. We discuss relations to other models. Next, the DOS and TLCF are calculated in Sec. III. In Sec. IV, we discuss the effect of the additional term and the DOS renormalization. Section V is devoted to discussion and conclusions.

II. SUPERSYMMETRIC NONLINEAR σ MODEL

A. Derivation

In this paper we treat the Hamiltonian in the form

$$H=\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}, \tag{1}$$

where W is an arbitrary matrix. This Hamiltonian possesses chiral symmetry, which means that the eigenvalues appear in pairs $\pm\epsilon_i$. The matrix W can be a rectangular matrix ($n\times m$) as well as a square one ($n=m$). $\nu=|n-m|$ is the topological number and is equal to the number of zero eigenvalues of the Hamiltonian. Here we consider $\nu=0$ and the extension to a finite ν will be discussed elsewhere. Making a unitary transformation, we have

$$H=\begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & -\Omega_1 \end{pmatrix}. \tag{2}$$

$\Omega_{1,2}$ are $n\times n$ Hermitian matrices. Treating these matrices as random ones, we can obtain the original chiral RMT [4]. Due to the chiral structure of the Hamiltonian, two random matrices couple in the single Hamiltonian and nontrivial correlations of the single Green function are expected. We restrict our discussion to the chiral unitary ensemble, which means $\Omega_{1,2}$ are arbitrary Hermitian matrices.

We consider a system written in field theoretical form as

$$\mathcal{H}=\int_{xy}\psi^\dagger(x)H(x,y)\psi(y), \tag{3}$$

where ψ is the fermionic field operator and $\int_x=\int d^dx$. The random Hamiltonian $H(x,y)$ has the chiral structure (2) and

$$\Omega_{1,2}(x,y)=\omega_{1,2}(x,y)a(|x-y|). \tag{4}$$

$\omega_{1,2}$ are random matrices and are averaged using the Gaussian integral

$$\langle\cdots\rangle=\int\mathcal{D}\omega_{1,2}(\cdots)\exp\left[-\frac{1}{\lambda^2}\int_{xy}[\omega_1(x,y)\omega_1(y,x)+\omega_2(x,y)\omega_2(y,x)]\right], \tag{5}$$

where λ is a free parameter. The function $a(r)$ represents a finite correlation of the Hamiltonian. We assume the range of the correlation, denoted by r_0, is large so that the saddle point approximation is applicable in the following calculation. In the limit $r\ll r_0$, $a(r)\sim 1$ and we have the fully Gaussian correlation. In the opposite limit $r\gg r_0$, we assume the correlation decays fast enough, e.g., $a(r)\sim\exp(-|r|/r_0)$.

This finite-range model is more realistic than chiral RMT in which all the matrix elements correlate with each other in the same way. The finite-range effect can be realized as a weak localization correction and a new energy scale $E_c=D/L^2$ (Thouless energy), where D is the diffusion constant and L the system length, comes into the analysis. Another interesting situation is when the decay of the matrix is power law. For a certain range of parameters this model reproduces the physics of the Anderson transition [15]. Extensions of the present work to the power-law case are discussed in Ref. [16].

We mention related work [17–21] in which similar nonlinear σ models were considered for systems with chiral symmetry. Our model is a simple generalization of models used in [17,20]. In other works, the random flux model [18], the random gauge field model [21], and the partially quenched chiral perturbation theory as the low-energy model of QCD [19] were considered. The derived nonlinear σ models differ from the standard diffusion model for nonchiral systems by symmetries of the matrix. Furthermore, an additional term was found in Refs. [17–19] although it was not found in Refs. [20,21]. Here we rederive the σ model and discuss relations to these models. In fact the additional term can exist and can be derived by a careful treatment of the massive mode integration. Although these models are different, we expect common low-energy properties. Our goal is to investigate them in the framework of the nonlinear σ model.

Let us derive the nonlinear σ model using the supersymmetry method. Our derivation is similar to that in Refs. [15,22]. We first define the generating function for the single Green function. Following Efetov's notation and conventions [6], we define it as $Z_1[J]=\int\mathcal{D}(\psi,\bar\psi)\exp(-\mathcal{L})$, with

$$\mathcal{L}=-i\int_{xy}\bar\psi(x)[\epsilon^+\delta(x-y)-H(x,y)+kJ(x)\delta(x-y)]\psi(y), \tag{6}$$

where $k=\mathrm{diag}(1,-1)$ operates in superspace, ψ is a four-component supervector, and $\bar\psi=\psi^\dagger$. The source field J is a 2×2 matrix in chiral space. We take the ensemble averaging to obtain

$$\mathcal{L} = -\frac{1}{4}\int_{xy} A(x,y)\mathrm{str}\,\tilde{\rho}(x)\tilde{\rho}(y) - i\int_x \bar{\psi}(x)[\epsilon^+ + kJ(x)]\psi(x), \tag{7}$$

where $A(x,y)=a^2(x-y)$, and

$$\tilde{\rho}(x) = \frac{1}{\sqrt{2}}[\rho(x) - \Sigma_x\rho(x)\Sigma_x],$$

$$\rho(x) = \Sigma_z^{1/2}\psi(x)\bar{\psi}(x)\Sigma_z^{1/2}. \tag{8}$$

$\Sigma_{x,z}$ are the Pauli matrices in chiral space. The Hubbard-Stratonovich field Q is introduced in the standard way. After integrations over ψ and $\bar{\psi}$, we have $\langle Z_1[J]\rangle = \int \mathcal{D}Q \times \exp(-F_1[J])$ with

$$F_1[J] = \frac{A_0}{2}\int_{xy}(A^{-1})(x,y)\mathrm{str}\,Q(x)Q(y) - \mathrm{str}\,\ln(\epsilon^+\Sigma_z + Jk\Sigma_z + i\lambda\sqrt{A_0}Q), \tag{9}$$

where $A_0 = \int_y A(x,y) \sim r_0^d$. Q is a 4×4 supermatrix and has the same symmetry as $\tilde{\rho}(x)$, which gives the condition $\{Q,\Sigma_x\}=0$.

We consider the saddle-point approximation. We are interested in the vicinity of the origin $\epsilon=0$ where chiral symmetry becomes important. At this point, the saddle-point equation gives $Q^2=1$ and the saddle-point manifold is obtained as $Q=T\Sigma_z\bar{T}$ where $\bar{T}$ is the inverse of T. Symmetries of the T matrix were considered in Ref. [23] and the explicit parametrization was obtained as

$$T = \sqrt{1-P^2} - iP, \quad P = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad t = \begin{pmatrix} a & \sigma \\ \rho & ib \end{pmatrix}, \tag{10}$$

where a,b are real variables and σ,ρ Grassmann variables. In addition, we must take into account the massive degrees of freedom which are not on the saddle-point manifold. Usually, in nonchiral systems, integrations of the massive degrees of freedom do not give any contribution. However, in the present case, the integrations give additional contributions written in terms of the massless modes. We can write the Q matrix as $Q=T(\Sigma_z+\delta Q)\bar{T}$ where δQ denotes the massive modes and changes the saddle point. Since the Q matrix anticommutes with Σ_x, the structure of δQ in chiral space is determined as

$$\delta Q = \begin{pmatrix} \delta q & 0 \\ 0 & -\delta q \end{pmatrix}, \tag{11}$$

where δq is a 2×2 supermatrix. This Q is substituted in the generating function and the functional F_1 is expanded in powers of δQ. We have

$$F_1[J] = F_1^{(0)}[J] + \tilde{F}_1^{(0)} + F_1^{(I)}, \tag{12}$$

where

$$F_1^{(0)}[J] = \frac{A_0}{2}\int_{xy} R(x,y)\mathrm{str}\,Q(x)Q(y) + \frac{i\pi\epsilon}{2V\Delta}\int_x \mathrm{str}\,\Sigma_z Q(x) + \frac{i\pi}{2V\Delta}\int_x \mathrm{str}\,J(x)k\Sigma_z Q(x),$$

$$\tilde{F}_1^{(0)} = \frac{A_0}{2}\int_{xy}[(A^{-1})(x,y) + \delta(x-y)A_0^{-1}]\mathrm{str}\,\delta Q(x)\delta Q(y),$$

$$F_1^{(I)} = \frac{A_0}{2}\int_{xy} R(x,y)\mathrm{str}\{2[\bar{T}(y)Q(x)T(y) - \Sigma_z]\delta Q(y) + T(x)\delta Q(x)\bar{T}(x)T(y)\delta Q(y)\bar{T}(y) - \delta Q(x)\delta Q(y) + \cdots\}. \tag{13}$$

$Q(x)=T(x)\Sigma_z\bar{T}(x)$, $R(x,y)=A^{-1}(x,y)-\delta(x-y)A_0^{-1}$, and $\Delta = \pi\lambda\sqrt{A_0}/2V$ (V is the system volume, and we put the lattice constant $a=1$) is the inverse of the DOS (mean level spacing) at $\epsilon=0$. $F_1^{(0)}[J]$ is independent of the massive modes, $\tilde{F}_1^{(0)}$ is the purely massive mode, and $F_1^{(I)}$ is the mixing term. Using the cumulant expansion and integrations of the massive modes we obtain $F_1 \sim F_1^{(0)}[J] + \langle F_1^{(I)}\rangle_{\tilde{F}_1^{(0)}}$ where

$$\langle F_1^{(I)}\rangle_{\tilde{F}_1^{(0)}} = \frac{1}{4}\int_{xy} R(x,y)[\mathrm{str}\,\bar{T}(y)T(x)\mathrm{str}\,\bar{T}(x)T(y) - \mathrm{str}\,\bar{T}(y)T(x)\Sigma_x\mathrm{str}\,\bar{T}(x)T(y)\Sigma_x]. \tag{14}$$

This calculation can be systematically done by using contraction rules derived in Appendix A. We neglected contributions that can be considered higher-order ones. The first term in the above equation is also neglected since the expansion does not include second order in P [see Eq. (10)]. We obtain

$$F_1 = \frac{A_0}{2}\int_{xy} R(x,y)\mathrm{str}\,Q(x)Q(y) - \frac{1}{4}\int_{xy} R(x,y)\mathrm{str}\,\bar{T}(y)T(x)\Sigma_x\mathrm{str}\,\bar{T}(x)T(y)\Sigma_x + \frac{i\pi\epsilon}{2\Delta V}\int_x \mathrm{str}\,\Sigma_z Q(x) + \frac{i\pi}{2\Delta V}\int_x \mathrm{str}\,J(x)k\Sigma_z Q(x). \tag{15}$$

The second term has a double-supertrace form and is not present in nonchiral systems. The crucial point is that the massive modes were parametrized as in Eq. (11). They have the structure Σ_z in chiral space. δQ in a form $\delta Q = \mathrm{diag}(\delta q_1, \delta q_2)$ would give the first term of $\langle F_1^{(I)}\rangle$ only, which is the case for nonchiral systems.

Using the gradient expansion, we obtain the final form of the σ model

$$F_1 = \frac{\pi D}{4\Delta V}\int \mathrm{str}(\nabla Q)^2 - \frac{\pi D_1}{16\Delta V}\int (\mathrm{str}\, Q\nabla Q\Sigma_x)^2 + \frac{i\pi\epsilon}{2\Delta V}\int \mathrm{str}\, Q\Sigma_z, \tag{16}$$

where we neglected the source term, $Q(x)=T(x)\Sigma_z\overline{T}(x)$ is a 4×4 supermatrix, and

$$\frac{\pi D}{\Delta V} = \frac{\int_r r^2 a^2(r)}{\int_r a^2(r)}, \quad \frac{\pi D_1}{\Delta V} = \frac{\int_r r^2 a^2(r)}{\left[\int_r a^2(r)\right]^2}. \tag{17}$$

Due to the relation $D=D_1\int_r a^2(r)\sim D_1 r_0^d$, the constant D_1 is smaller than D by the factor $1/r_0^d$ and the second term in Eq. (16) can be neglected. However, it can be important when the quantum effect is taken into account by the renormalization group method. It is discussed in Sec. IV.

The generating function Z_1 is used only for a single Green function. It is straightforward to extend the calculation to the case of products of Green functions. The generating function for the product of the retarded Green function $\langle \mathrm{tr}\, G^{(R)} \times(\epsilon_1)\mathrm{tr}\, G^{(R)}(\epsilon_2)\rangle$ is defined as

$$Z_2[J] = \int \mathcal{D}(\psi,\overline{\psi})\exp\left[i\int \overline{\psi}(\hat{\epsilon}^+ - H + kJ)\psi\right], \tag{18}$$

where $\psi, \overline{\psi}$ are eight-component supervectors. $\hat{\epsilon} = \mathrm{diag}(\epsilon_1,\epsilon_2)$ is the matrix in "two-point" space. In chiral symmetric systems, the identity $\mathrm{tr}G^{(A)}(\epsilon)=-\mathrm{tr}G^{(R)}(-\epsilon)$ holds and the generating function for the advanced Green function can be found from Z_2. Repeating the calculation in a similar way, we find the σ model $\langle Z_2\rangle = \int\mathcal{D}Q\,\exp(-F_2)$ with

$$F_2 = \frac{\pi D}{4\Delta V}\int \mathrm{str}(\nabla Q)^2 - \frac{\pi D_1}{32\Delta V}\int [(\mathrm{str}\, Q\nabla Q\Sigma_x)^2 + (\mathrm{str}\,\Lambda Q\nabla Q\Sigma_x)^2] + \frac{i\pi}{2\Delta V}\int \mathrm{str}\,\hat{\epsilon}\Sigma_z Q, \tag{19}$$

where $Q=T\Sigma_z\overline{T}$ is an 8×8 supermatrix and $\Lambda=\mathrm{diag}(1,-1)$ in two-point space.

B. Comparison with other models

Our derived nonlinear σ model is equivalent to the models in Refs. [14,20] except for the presence of the double-trace term. The reason why that term was absent in Refs. [14,20] is that the massive mode integration was not taken carefully.

In order to compare our result with the models in Refs. [17,18] we use the Q-matrix parametrization

$$Q = \Sigma_z[(1-P^2)^{1/2} + iP]^2, \quad P = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \tag{20}$$

where t is a 2×2 supermatrix. The random flux model in Ref. [18] is mapped onto the effective action

$$S_{RF} = -\frac{2}{b}\int \mathrm{str}\ \nabla T^{-1}\nabla T - \frac{1}{c}\int (\mathrm{str}\, T^{-1}\nabla T)^2 - \frac{2i\omega}{b}\int \mathrm{str}(T+T^{-1}), \tag{21}$$

where $T\in \mathrm{GL}(n|n)$. This model is reduced to our model by using the parametrization

$$T = [t + (1+t^2)^{1/2}]^2, \tag{22}$$

and putting $n=1$. The "flavor" degrees of freedom n represent different species of electrons and are not important for the present problem. We note that different notation and conventions are used in this expression. In contrast with our definition of supermathematics [6], the definition in Ref. [24] was used in Eq. (21), which explains the difference in appearance between Eqs. (16) and (21).

It is worthwhile to mention the relation of the coupling constants b and c. The authors in Ref. [18] found the relation $b\sim c/N$ where N are the "color" degrees of freedom. N must be large in order to justify the saddle-point approximation. Thus the second term in Eq. (21) is small compared with the first term. This is precisely what we found, and the correlation length r_0 corresponds to N. We also note that we neglected the topological term coming from the boundary condition [18]. Such a term is expected to be derived in our model by considering a finite topological number ν, and it will be discussed elsewhere.

In a similar way, our result is compared with Gade's replica σ model based on the sublattice models [17]

$$S_G = \frac{2}{b}\int \mathrm{tr}\ \nabla(Z+W)\nabla(Z-W) - \frac{1}{c}\int [\mathrm{tr}(W\nabla Z - Z\nabla W)]^2 - \frac{4i\omega}{b}\int \mathrm{tr}\, W, \tag{23}$$

where Z is a matrix with some symmetry and $W=(1+Z^2)^{1/2}$. The parametrization

$$Z = 2t(1+t^2)^{1/2}, \quad W = 1 + 2t^2 \tag{24}$$

is used to find a formal agreement with our model. We note that Gade's model was obtained by using the replica method and the structure of the matrix t is different from ours. However, we show in the following that, at least in the perturbative regime, both calculations give the same result. It is known that the replica and supersymmetry methods give the same perturbative result for the same symmetry class.

The relation of the coupling constants b and c was not discussed in Ref. [17]. It is not clear what is the large parameter in the model to justify the saddle-point approximation. It is expected that introduction of such a parameter leads to a similar relation just as in other calculations.

Both works [17,18] did not use the Q-matrix representation. It has been used in traditional σ models and is useful for comparison of models and for formulation of perturbative and nonperturbative calculations as we demonstrate below. It is also important to find gauge invariance of the model.

III. LEVEL CORRELATION FUNCTIONS

In this section, we calculate the DOS and TLCF by using the nonlinear σ models derived in the previous section. We neglect the double-trace term contribution and put $D_1=0$. This is because D_1 is smaller than D by the factor $1/r_0$ at the classical level. The effect of the double-trace term is discussed in Sec. IV.

We write down the DOS and TLCF in a functional integral form. The DOS is given by

$$\langle\rho(\epsilon)\rangle = \frac{1}{4\Delta V}\mathrm{Re}\int \mathcal{D}Q\left[\int_x \mathrm{str}\, k\Sigma_z Q(x)\right]e^{-F_1}, \qquad (25)$$

where F_1 is given by Eq. (16) (we put $D_1=0$) and Q is a 4×4 supermatrix. The TLCF is

$$\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle = \frac{1}{4}[W(\epsilon_1,\epsilon_2)+W(\epsilon_1,-\epsilon_2)+W(-\epsilon_1,\epsilon_2) + W(-\epsilon_1,-\epsilon_2)], \qquad (26)$$

$$W(\epsilon_1,\epsilon_2) = \frac{1}{16\Delta^2V^2}\int \mathcal{D}Q\left[\int_x \mathrm{str}\, k\Lambda_1\Sigma_z Q(x)\right] \times\left[\int_y \mathrm{str}\, k\Lambda_2\Sigma_z Q(y)\right]e^{-F_2}, \qquad (27)$$

where F_2 is given by Eq. (19), Q is an 8×8 supermatrix, and $\Lambda_{1,2}=(1\pm\Lambda)/2$. In the following we use the connected part of the TLCF $\langle\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle\rangle=\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle-\langle\rho(\epsilon_1)\rangle\langle\rho(\epsilon_2)\rangle$.

A. Summary of the result

Before entering into the detailed analysis, we give an outline of the derivation and the result for reference. For perturbation theory, the Q matrix is expanded in powers of the P matrix:

$$Q(x) = \Sigma_z\frac{1+iP}{1-iP} = \Sigma_z(1+2iP-2P^2+\cdots). \qquad (28)$$

Correspondingly, the result is expressed by using the expansion of the diffusion propagator [6]

$$\Pi(q,\epsilon) = \frac{\Delta}{2\pi}\frac{1}{Dq^2-i\epsilon}. \qquad (29)$$

The expansion parameter is $1/g$ where $g=\pi E_c/\Delta=\pi D/\Delta L^2$ is the dimensionless conductance. It does not appear in the zero-mode sector of the propagator ($q=0$) and the expansion is not justified. Actually, treating the zero mode exactly (nonperturbatively), and neglecting other nonzero modes, we can obtain the ergodic result. The Q matrix for the zero mode is written as

$$Q = T\Sigma_z\overline{T}, \qquad (30)$$

where T is independent of the spatial coordinate and its explicit parametrization is given in the following. In order to incorporate the zero and nonzero modes into the analysis we should use the parametrization

$$Q(x) = T\widetilde{Q}(x)\overline{T}. \qquad (31)$$

$\widetilde{Q}$ parametrizes the nonzero modes and is expanded in powers of the P matrix as in Eq. (28). The zero mode $Q=T\Sigma_z\overline{T}$ is treated nonperturbatively so that the ergodic result is obtained. This parametrization is reminiscent of the renormalization group calculation (see, e.g., Ref. [6]) and was used by KM. They considered integrations of the nonzero modes first and found corrections to the ergodic result. For a technical reason the result was applicable only to the domain $z\ll g$ where $z=\pi\epsilon/\Delta$ is the scaled energy variable. Here we consider the zero-mode integration first and then integrate the nonzero modes. This method allows us to consider the domain $z\gg 1$ discussed by AA. For comparison, we present the KM method in Sec. III C.

The zero-mode model is equivalent to chiral RMT. This ergodic limit can be obtained by putting $g=\infty$ in the above functional integral form. The result is scaled by the mean level spacing Δ to give

$$\rho_1(z) = \Delta\langle\rho(\epsilon=\Delta z/\pi)\rangle = \rho_1^{(0)}(z), \qquad (32)$$

$$\rho_2(z_1,z_2) = \Delta^2\langle\langle\rho(\epsilon_1=\Delta z_1/\pi)\rho(\epsilon_2=\Delta z_2/\pi)\rangle\rangle = -K^2(z_1,z_2), \qquad (33)$$

where

$$\rho_1^{(0)}(z) = \frac{\pi z}{2}[J_0^2(z)+J_1^2(z)],$$

$$K(z_1,z_2) = \frac{\pi\sqrt{z_1z_2}}{z_1^2-z_2^2}[z_1J_1(z_1)J_0(z_2)-z_2J_0(z_1)J_1(z_2)]. \qquad (34)$$

This result does not depend on any parameter and is universal. It was obtained in Ref. [25] by using the orthogonal polynomial method and in Ref. [23] using the supersymmetry method.

How is it changed if we include the nonzero modes? If we treat all the modes perturbatively, the result is expressed by the diffusion propagator. The expansion (28) is used to give

$$\langle\rho(\epsilon)\rangle \sim \frac{1}{\Delta}\left[1+\frac{1}{2}\mathrm{Re}\left(\sum_q \Pi(q,\epsilon)\right)^2\right], \qquad (35)$$

$$\langle\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle\rangle \sim \frac{1}{2\Delta^2}\mathrm{Re}\sum_q[\Pi^2(q,(\epsilon_1+\epsilon_2)/2) + \Pi^2(q,(\epsilon_1-\epsilon_2)/2)]. \qquad (36)$$

This expression includes the zero mode and is justified for $g\gg 1$ and $z\gg 1$. The zero-mode contribution gives the asymptotic form of the ergodic result as was shown in Ref. [14].

Before discussing the exact treatment of the zero mode we must mention the effect of the renormalization of the mean level spacing. The quantity Δ was introduced as the mean level spacing at $g=\infty$ and $z=\infty$. For traditional symmetry classes, it remains unchanged even if we include the nonzero modes (finite-g effect), which is a consequence of the particle conservation law. However, this is not the case in chiral systems. For finite g, the nonzero modes contribute to Δ, which means that the DOS is renormalized as was discussed in Ref. [17]. Referring to Eq. (35), we define the renormalized mean level spacing

$$\frac{1}{\tilde{\Delta}} \sim \frac{1}{\Delta}\left[1+\frac{1}{2}\mathrm{Re}\left(\sum_{q\neq 0}\Pi(q,\epsilon)\right)^2+\cdots\right]. \qquad (37)$$

Note that the zero mode is excluded in this expression. Contributions of the zero mode are totally different from those of other modes. The nonzero modes determine the macroscopic behavior of the DOS $1/\tilde{\Delta}$, while the zero mode determines the universal microscopic behavior after scaling in terms of $\tilde{\Delta}$.

A naive calculation shows that $\tilde{\Delta}$ is divergent in some cases and should be renormalized to a finite value using a regularization. We are interested in the microscopic behavior after the mean level spacing is scaled out. The effect of nonzero modes in the microscopic domain is present even after the scaling and we discuss it in the following.

We turn to the main results in this section. We use the parametrization (31) to treat the zero and nonzero modes separately. The zero mode is parametrized so that the ergodic results (32) and (33) are reproduced and the nonzero modes are treated perturbatively. The domain $z \ll g$ was first considered by KM for nonchiral systems and we call it KM's domain. Up to second order in $1/g$, the DOS in KM's domain is given by

$$\rho_1(z)=\tilde{\Delta}\langle\rho(\epsilon=\tilde{\Delta}z/\pi)\rangle \sim \left[1+\frac{a_d}{8g^2}\left(2z\frac{d}{dz}+z^2\frac{d^2}{dz^2}\right)\right]\rho_1^{(0)}(z). \qquad (38)$$

a_d is the momentum integration

$$a_d=\frac{1}{8\pi^4}\sum_{n\geqslant 0,n^2\neq 0}\left(\frac{1}{n^2}\right)^2. \qquad (39)$$

We used the periodic boundary condition. The TLCF is

$$\begin{aligned}\rho_2(z_1,z_2)&=\tilde{\Delta}^2\langle\langle\rho(\epsilon_1=\tilde{\Delta}z_1/\pi)\rho(\epsilon_2=\tilde{\Delta}z_2/\pi)\rangle\rangle\\ &\sim -\left\{\left[1+\frac{a_d}{8g^2}\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}\right)+\frac{a_d}{8g^2}\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}\right)^2\right]K(z_1,z_2)\right\}^2. \end{aligned} \qquad (40)$$

The result was scaled by the renormalized mean level spacing (37). The calculation of the DOS for chiral systems has been done in Ref. [14] but the renormalized mean level spacing was not introduced. It leads to a different conclusion on level statistics as we discuss in Secs. III C and V.

We now consider the AA domain $z \gg 1$, $g \gg 1$. The scaled DOS is given by

$$\rho_1(z) \sim 1-\frac{\cos 2z}{2z}\mathcal{D}(z)+\frac{1}{8z^2}, \qquad (41)$$

where $\mathcal{D}(z)$ is the spectral determinant

$$\mathcal{D}(z)=\prod_{q\geqslant 0,q^2\neq 0}\frac{(Dq^2)^2}{(Dq^2)^2+\epsilon^2}=\prod_{n\geqslant 0,n^2\neq 0}\frac{g^2(4\pi^2n^2)^2}{g^2(4\pi^2n^2)^2+z^2}. \qquad (42)$$

The TLCF is

$$\begin{aligned}\rho_2(z_1,z_2) \sim{}& \frac{1}{2}\mathrm{Re}\sum_{q^2\neq 0}(\Pi_+^2+\Pi_-^2)\\ &+\frac{\sin 2z_1}{2z_1}\mathcal{D}_1\mathrm{Im}\sum_{q^2\neq 0}(\Pi_+ + \Pi_-)\\ &+\frac{\sin 2z_2}{2z_2}\mathcal{D}_2\mathrm{Im}\sum_{q^2\neq 0}(\Pi_+ - \Pi_-)\\ &+\frac{1}{8z_1z_2}[\mathcal{D}_1\mathcal{D}_2(\mathcal{D}_+^2\mathcal{D}_-^{-2}-1)\cos 2(z_1+z_2)\\ &+\mathcal{D}_1\mathcal{D}_2(\mathcal{D}_-^2\mathcal{D}_+^{-2}-1)\cos 2(z_1-z_2)]\\ &-\frac{1}{2(z_1+z_2)^2}[1+\mathcal{D}_1\mathcal{D}_2\mathcal{D}_+^2\mathcal{D}_-^{-2}\cos 2(z_1+z_2)]\\ &-\frac{1}{2(z_1-z_2)^2}[1-\mathcal{D}_1\mathcal{D}_2\mathcal{D}_-^2\mathcal{D}_+^{-2}\cos 2(z_1-z_2)]\\ &+\frac{1}{z_1^2-z_2^2}(\mathcal{D}_1\sin 2z_1-\mathcal{D}_2\sin 2z_2), \end{aligned} \qquad (43)$$

where $\mathcal{D}_{1,2}=\mathcal{D}(z_{1,2})$, $\mathcal{D}_\pm=\mathcal{D}((z_1\pm z_2)/2)$, and $\Pi_{1,2}=\Pi(q,\epsilon_{1,2})$, $\Pi_\pm=\Pi(q,(\epsilon_1\pm\epsilon_2)/2)$. The result is expressed using the spectral determinant as the AA result [9]. Equation (41) can be interpreted as follows. Consider the asymptotic form of the ergodic result (32),

$$\rho_1(z) \sim 1-\frac{\cos 2z}{2z}+\frac{1}{8z^2}+\cdots. \qquad (44)$$

Then, including the spectral determinant in the oscillating term, one finds Eq. (41). Equation (43) is more complicated, but we can see that the ergodic limit gives the asymptotic form of the exact result (33). While standard perturbation theory gives nonoscillating terms, expansions around two saddle points [9] are required to get oscillating terms.

We emphasize that Eqs. (38), (40), (41), and (43) are the main results in this section. They have the following properties.

Common domain $1 \ll z \ll g$. The KM and AA results have a common domain $1 \ll z \ll g$ where the asymptotic expansion of the Bessel function and the expansion of the spectral determinant in z/g can be used. In this domain, the DOS and TLCF are approximated as

$$\rho_1(z) \sim 1-\frac{\cos 2z}{2z}+\frac{1}{8z^2}+\frac{a_d}{4g^2}z\cos 2z, \qquad (45)$$

$$\rho_2(z_1,z_2) \sim -\left\{\frac{\sin(z_1-z_2)}{z_1-z_2} - \frac{\cos(z_1+z_2)}{z_1+z_2} + \frac{a_d}{8g^2}(z_1 + z_2)\cos(z_1+z_2) - \frac{a_d}{8g^2}(z_1-z_2)\sin(z_1-z_2)\right\}^2. \quad (46)$$

Small z. At small energies, the expansion of the Bessel function in z is used in Eqs. (38) and (40) to give

$$\rho_1(z) \sim \frac{\pi z}{2}\left(1 + \frac{a_d}{4g^2}\right), \quad (47)$$

$$\rho_2(z_1,z_2) \sim -\frac{\pi^2 z_1 z_2}{4}\left(1 + \frac{a_d}{2g^2}\right). \quad (48)$$

These results show that level repulsion at the origin weakens, which is consistent with the intuitive picture.

Unitary limit. Taking z, $z_1+z_2\to\infty$, we obtain the unitary limit as $\rho_1(z)\to 1$ and

$$R(z_1,z_2) = 1 + \frac{\rho_2(z_1,z_2)}{\rho_1(z_1)\rho_1(z_2)} \to 1 + \frac{1}{2}\mathrm{Re}\sum_{q^2\neq 0}\Pi_-^2 - \frac{1}{2(z_1-z_2)^2} + \frac{\cos 2(z_1-z_2)}{2(z_1-z_2)^2}\mathcal{D}\left(\frac{z_1-z_2}{2}\right). \quad (49)$$

This result is consistent with the AA result [9] for the unitary class. We note the relations $\Pi(q,\epsilon/2;g)=2\Pi(q,\epsilon;2g)$ and $\mathcal{D}(z/2;g)=\mathcal{D}(z;2g)$. The coefficient 2 in front of g originates from chiral symmetry. Comparing our σ model (19) with the model for unitary symmetry [6], we see the size of the Q matrix is doubled due to chiral symmetry.

$z_1=z_2$. The relation $\rho_2(z,z)=-\rho_1^2(z)$ holds for arbitrary g. It can be used to derive the DOS from the TLCF.

B. Density of states

1. Perturbative calculation

Now we go into details of the calculation of the DOS (25). The perturbative calculation for nonzero modes is considered using the expansion of the Q matrix in P as Eq. (28). The P matrix is parametrized for the chiral unitary class as

$$P = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad t = \begin{pmatrix} a & \sigma \\ \rho & ib \end{pmatrix}, \quad (50)$$

where a, b are real variables, and σ, ρ Grassmann ones. The measure of this parametrization is normalized to unity. We define the average

$$\langle\cdots\rangle = \int \mathcal{D}Q(\cdots)e^{-F_1^{(0)}},$$

$$F_1^{(0)} = \frac{\pi D}{\Delta V}\int_x \mathrm{str}(\nabla P)^2 - \frac{i\pi\epsilon}{\Delta V}\int_x \mathrm{str}\, P^2, \quad (51)$$

where $F_1^{(0)}$ is the second order part of F_1. Performing expansions in P as

$$\langle\rho(\epsilon)\rangle \sim \frac{1}{\Delta}\mathrm{Re}\left(1 - \frac{1}{2V}\int_x\langle \mathrm{str}\, kP^2\rangle + \frac{1}{2V}\int_x\langle\mathrm{str}\, kP^4\rangle + \cdots\right) \quad (52)$$

and using the contraction rules derived in Appendix A as Eq. (A6), we obtain the result (35).

As we emphasized in the previous subsection, this perturbative calculation of the nonzero modes suggests that the mean level spacing $\tilde{\Delta}$ is renormalized as Eq. (37). The exact definition of $\tilde{\Delta}$ can be written as

$$\frac{1}{\tilde{\Delta}} = \frac{1}{\Delta}\int \mathcal{D}\tilde{Q}\left[\frac{1}{4V}\int_x \mathrm{str}\, k\Sigma_z\tilde{Q}(x)\right]e^{-F_1[\tilde{Q}]}. \quad (53)$$

Thus effect of the self-interacting diffusion bubble is renormalized to the mean level spacing. It corresponds to imposing the constraint $\langle\tilde{Q}\rangle_{F_1}=\Sigma_z$. In Sec. III C we give a detailed analysis using the KM method.

2. Ergodic limit

At the ergodic limit $g\to\infty$, spatial dependence of the Q matrix is neglected and the DOS is reduced to the form

$$\rho_1(z) = \frac{1}{4}\int \mathcal{D}Q\, \mathrm{str}\, k\Sigma_z Q\, \exp\left(-\frac{iz}{2}\mathrm{str}\,\Sigma_z Q\right). \quad (54)$$

Following Ref. [23], we parametrize the Q matrix as

$$Q = T\Sigma_z\overline{T}, \quad T = UT_0\overline{U},$$

$$T_0 = \begin{pmatrix} \cos\frac{\hat{\theta}}{2} & -i\sin\frac{\hat{\theta}}{2} \\ -i\sin\frac{\hat{\theta}}{2} & \cos\frac{\hat{\theta}}{2}\end{pmatrix}, \quad \hat{\theta} = \begin{pmatrix}\theta_F & 0 \\ 0 & i\theta_B\end{pmatrix},$$

$$U = \begin{pmatrix} u & 0 \\ 0 & u\end{pmatrix}, \quad u = \exp\begin{pmatrix} 0 & \xi \\ \eta & 0\end{pmatrix}, \quad (55)$$

where $-\pi\leqslant\theta_F\leqslant\pi$ and $0\leqslant\theta_B\leqslant\infty$. The measure is given by

$$\mathcal{D}Q = d\theta_B d\theta_F d\xi\, d\eta\frac{1}{2\pi} \times \frac{\cosh\theta_B\cos\theta_F - 1 - i\sinh\theta_B\sin\theta_F}{(\cosh\theta_B - \cos\theta_F)^2}. \quad (56)$$

We note that the compact (noncompact) variable θ_F (θ_B) is used for the fermion-fermion (boson-boson) block [6]. Substituting this parametrization into Eq. (54) and integrating the Grassmann variables, we find

$$\rho_1(z) = 1 + \mathrm{Im}\int_z^\infty dt\int_0^\infty d\theta_B\int_0^\pi d\theta_F\frac{1}{\pi}(\cosh\theta_B\cos\theta_F - 1) \times e^{it^+(\cosh\theta_B-\cos\theta_F)} = 1 - \frac{\pi}{2}\int_z^\infty dt[J_0^2(t) - J_1^2(t)]. \quad (57)$$

Here we introduced the auxiliary variable $t^+=t+i0$ and assumed $z>0(t>0)$. For the Bessel function, we used integral representations

$$J_0(z) = \mathrm{Re}\frac{-2i}{\pi}\int_0^\infty d\theta_B e^{iz^+\cosh\theta_B} \tag{58}$$

for the noncompact variable and

$$J_0(z) = \frac{1}{\pi}\int_0^\pi d\theta_F e^{iz\cos\theta_F}, \tag{59}$$

for the compact variable. J_1 is given by $J_1(z)=-J_0'(z)$. Integrating the variable t, we obtain Eq. (32).

The asymptotic form at $z\gg 1$ is given by Eq. (44). This cannot be obtained by standard perturbation theory which gives only nonoscillating terms, the first and third terms in Eq. (44). The oscillating second term can be obtained by taking into account two saddle points for integrals of $\theta_{B,F}$ in Eq. (57). In addition to the "standard saddle point" $(\theta_B,\theta_F)=(0,0)$ we have another "supersymmetry-breaking saddle point" $(0,\pi)$. We note that the point (0,0) corresponds to $Q=\Sigma_z$ and $(0,\pi)$ to $Q=-k\Sigma_z$. This is precisely the idea of the calculation in Ref. [9]. Taking into account fluctuations around these points, we can obtain the desired result.

In fact this idea is used to find the correct asymptotics of the Bessel function. The noncompact representation (58) is used for θ_B and has the saddle point $\theta_B=0$. The compact representation (59) for θ_F has the saddle points $\theta_F=0,\pi$. Expanding around these saddle points, respectively, we have

$$J_0(z) \sim \sqrt{\frac{1}{\pi z}}\left[\left(1-\frac{1}{8z}+\cdots\right)\cos z + \left(1+\frac{1}{8z}+\cdots\right)\sin z\right]. \tag{60}$$

It is interesting to note that the expansion around a single saddle point is required for the noncompact representation (58) and two points for the compact representation (59). When Eq. (58) is deformed to Eq. (59) the single point $\theta_B=0$ splits into the two points $\theta_F=0,\pi$. It can be shown by considering the deformation of the integral contour used in Ref. [23]. We find

$$\frac{-2i}{\pi}\int_0^\infty d\theta\, e^{iz^+\cosh\theta} = \frac{2}{\pi}\int_0^{\pi/2} d\theta\, e^{iz\cos\theta} - \frac{2i}{\pi}\int_0^\infty d\theta\, e^{-z\sinh\theta}. \tag{61}$$

This representation is known as the Hankel function $H_0=J_0+iN_0$. Taking the real part, we obtain

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi/2} d\theta\, e^{iz\cos\theta} + \frac{1}{\pi}\int_0^{\pi/2} d\theta\, e^{-iz\cos\theta}. \tag{62}$$

This expression is reduced to Eq. (59) by changing the variable $\theta\to\pi-\theta$ in the second term. Thus the point 0 in the second term is changed to π. Note that the real part of the integral is taken in the noncompact representation (58), which gives the second term.

This method, taking into account a set of nontrivial saddle points, is the main idea of the nonperturbative calculation. It produces the exact result for the unitary class and the asymptotic ones for the orthogonal and symplectic classes [9]. It has been used even for the replica [26] and Keldysh [27] σ models. For chiral symmetric systems at the ergodic limit, a similar technique has been used in Ref. [28] to find the asymptotic result (44). In the following, we examine how the effect of nonzero modes is incorporated into the asymptotic form.

3. Integration of the zero mode

We write the Q matrix as Eq. (31) and use the parametrizations (28) and (50) for $\tilde{Q}$, and (55) for T. It is slightly modified as

$$Q(x) = UT_0\overline{U}\tilde{Q}(x)U\overline{T_0}\,\overline{U} \to UT_0\tilde{Q}(x)\overline{T_0}\,\overline{U}. \tag{63}$$

As a result, F_1 becomes independent of the Grassmann variables of the zero mode. The preexponential term in Eq. (25) is written explicitly using the Grassmann variables as

$$\mathrm{str}\,k\Sigma_z Q(x) \to \mathrm{str}\,k\Sigma_z T_0\tilde{Q}(x)\overline{T_0} + 2\xi\eta\,\mathrm{str}\,\Sigma_z T_0\tilde{Q}(x)\overline{T_0}. \tag{64}$$

We neglected contributions that vanish after integrations over ξ and η. The first term does not include those variables and we can put $T_0=1$ for the integrations. The second term is also easily integrated out and we thus have $\langle\rho(\epsilon)\rangle=\langle\rho(\epsilon)\rangle_1+\langle\rho(\epsilon)\rangle_2$ where

$$\langle\rho(\epsilon)\rangle_1 = \frac{1}{4\Delta V}\mathrm{Re}\int \mathcal{D}\tilde{Q}\mathcal{J}[\tilde{Q}]\left[\int \mathrm{str}\,k\Sigma_z\tilde{Q}\right]e^{-F_1[\tilde{Q}]},$$

$$\langle\rho(\epsilon)\rangle_2 = \frac{1}{2\pi\Delta}\mathrm{Re}\int_0^\infty d\theta_B\int_{-\pi}^{\pi} d\theta_F \times\frac{(\cosh\theta_B\cos\theta_F - 1 + i\sinh\theta_B\sin\theta_F)}{(\cosh\theta_B-\cos\theta_F)^2}\times I(\epsilon,\theta_B,\theta_F),$$

$$I(\epsilon,\theta_B,\theta_F) = -i\frac{\partial}{\partial z}\int \mathcal{D}\tilde{Q}\mathcal{J}[\tilde{Q}]\exp\left[-\frac{\pi D}{4\Delta V}\int \mathrm{str}(\nabla\tilde{Q})^2 - \frac{i\pi\epsilon}{2\Delta V}\int \mathrm{str}\,\Sigma_z T_0\tilde{Q}\overline{T_0}\right]. \tag{65}$$

$\langle\rho(\epsilon)\rangle_1$ gives the perturbative result (35) without the zero-mode contribution and is equal to the inverse of the renormalized mean level spacing $1/\tilde{\Delta}$. $\langle\rho(\epsilon)\rangle_2$ includes the ergodic result and is nonperturbative.

We note that the Jacobian $\mathcal{J}[\tilde{Q}]$ contribution exists in the present parametrization (31). It depends on the nonzero modes only and can be written as

$$\mathcal{J}[\tilde{Q}] = \exp\left[\frac{1}{4V}\int (\mathrm{str}\,P\Sigma_x)^2 + O(P^4)\right]. \tag{66}$$

This contribution changes the renormalized mean level spacing slightly and the scaled DOS $\rho_1(z)$ is not changed in our approximation. For this reason, we neglect this contribution in the present section. It is treated in Sec. IV when we discuss the DOS renormalization.

Let us turn to the calculation of $\langle\rho(\epsilon)\rangle_2$. The kinetic term in F_1 does not include the zero mode and is expanded in powers of P. The second term in F_1 (and the preexponential term) is expanded as

$$-\frac{1}{2}\int \mathrm{str}\,\Sigma_z T_0\widetilde{Q}\overline{T_0} = V(\cosh\theta_B - \cos\theta_F) + \int[\mathrm{str}(k_F P^2)\cos\theta_F + \mathrm{str}(k_B P^2)\cosh\theta_B] - \int[\mathrm{str}(k_F\Sigma_x P^3)\sin\theta_F - i\,\mathrm{str}(k_B\Sigma_x P^3)\sinh\theta_B] + \cdots, \quad (67)$$

where $k_{F,B}=(1\pm k)/2$. In the following calculation we neglect odd terms in the P matrix. Their contributions give $1/g^3$ corrections at most. Another reason to neglect them is that they involve a factor $\sin\hat{\theta}$ which goes to zero at the saddle points $\theta_F=0,\ \pi$ and $\theta_B=0$. Using this approximation, we find the simplified expression

$$I(\epsilon,\theta_B,\theta_F) \sim -i\frac{\partial}{\partial z}e^{iz(\lambda_B-\lambda_F)}\langle e^{iz\lambda_B A_B+iz\lambda_F A_F}\rangle_{\mathrm{kin}},$$

$$A_{F,B}[\widetilde{Q}] = -\frac{1}{2V}\int \mathrm{str}\,k_{F,B}\Sigma_z[\widetilde{Q}-\Sigma_z],$$

$$\langle\cdots\rangle_{\mathrm{kin}} = \int \mathcal{D}\widetilde{Q}(\cdots)e^{-F_{\mathrm{kin}}},$$

$$F_{\mathrm{kin}} = \frac{\pi D}{4\Delta V}\int \mathrm{str}(\nabla\widetilde{Q})^2, \quad (68)$$

where $z=\pi\epsilon/\Delta$, $\lambda_B=\cosh\theta_B$, and $\lambda_F=\cos\theta_F$. $A_{F,B}[\widetilde{Q}]$ include even powers in P. Introducing the auxiliary variable t, we obtain

$$\langle\rho(\epsilon)\rangle_2 = \frac{1}{\pi\Delta}\mathrm{Im}\int_z^\infty dt\int_0^\infty d\theta_B\int_0^\pi d\theta_F(\lambda_B\lambda_F-1)e^{it^+(\lambda_B-\lambda_F)}\left[1-(t-z)\frac{\partial}{\partial z}\right]\langle e^{iz\lambda_B A_B+iz\lambda_F A_F}\rangle_{\mathrm{kin}}. \quad (69)$$

Now the problem is how integrations of the variables $\theta_{B,F}$ are performed. They can be done by noting that the variable t in the exponential is shifted to $t^+ + zA_B$ or $t - zA_F$ compared with the ergodic limit. For the fermion part θ_F, there is no convergence problem and the Bessel function is derived. It is also the case for the boson part θ_B since the convergence problem does not arise for the part including Grassmann variables and the other parts are real. The only difference is that we cannot take the real part for the expression after integration of θ_B since the argument $t+zA_B$ includes Grassmann variables. We get the Hankel function $H_0=J_0+iN_0$ instead of the Bessel function J_0 [see Eq. (61)]. However, the imaginary part iN_0 does not contribute to the final result since the functional A_B is reduced to a real function in the end. This is valid in our approximation keeping contributions up to second order in $1/g$. Thus we neglect the imaginary part and obtain

$$\langle\rho(\epsilon)\rangle_2 \sim \frac{\pi}{2\Delta}\mathrm{Re}\frac{d}{dz}\int_z^\infty dt(t-z)\langle J_0(t+zA_B)J_0(t-zA_F) - J_1(t+zA_B)J_1(t-zA_F)\rangle_{\mathrm{kin}}. \quad (70)$$

The ergodic limit $g=\infty$ can be found easily by putting $A_{F,B}[\widetilde{Q}]=0$. We note again that this equation was obtained by neglecting contributions including $\sin\theta_F$ or $\sinh\theta_B$. This approximation is valid up to second order in $1/g$. It still remains to carry out integrations over the nonzero modes. In the following we consider two limiting cases.

4. KM's domain ($z\ll g$)

The case $z\ll g$ can be considered using KM's method [8]. For chiral systems, it was considered in Ref. [14]. In our method, the Bessel functions in Eq. (70) are expanded in powers of $zA_{F,B}\sim O(z/g)$ to find

$$\langle\rho(\epsilon)\rangle_2 \sim \frac{1}{\Delta}\mathrm{Re}\left[1+\frac{1}{2}\langle A_B-A_F\rangle_{\mathrm{kin}}\frac{d}{dz}z+\frac{1}{8}\langle(A_B-A_F)^2\rangle_{\mathrm{kin}}\frac{d}{dz}z^2\frac{d}{dz}\right][\rho_1^{(0)}(z)-1]. \quad (71)$$

Combining with the perturbative contribution

$$\langle\rho(\epsilon)\rangle_1 \sim \frac{1}{\Delta}\mathrm{Re}\left[1+\frac{1}{2}\langle A_B-A_F\rangle_{\mathrm{kin}}+\cdots\right], \quad (72)$$

we find

$$\langle\rho(\epsilon)\rangle \sim \frac{1}{\Delta}\mathrm{Re}\left[1+\frac{1}{2}\langle A_B-A_F\rangle_{\mathrm{kin}}\frac{d}{dz}z+\frac{1}{8}\langle(A_B-A_F)^2\rangle_{\mathrm{kin}}\frac{d}{dz}z^2\frac{d}{dz}\right]\rho_1^{(0)}(z). \quad (73)$$

Up to here the DOS is scaled in terms of the bare mean level spacing Δ. We introduce the renormalized mean level spacing as $1/\widetilde{\Delta}=\langle\rho(\epsilon)\rangle_1$. Defining the energy variable as $\widetilde{z}=\pi\epsilon/\widetilde{\Delta}$, we use the transformation formula for a function $f(z)$

$$f(z) = \left[1+\left(\frac{\widetilde{\Delta}}{\Delta}-1\right)\widetilde{z}\frac{d}{d\widetilde{z}}+\cdots\right]f(\widetilde{z}) \sim \left[1-\frac{1}{2}\langle A_B-A_F\rangle_{\mathrm{kin}}\widetilde{z}\frac{d}{d\widetilde{z}}+\cdots\right]f(\widetilde{z}). \quad (74)$$

It is applied to Eq. (73) to find

$$\rho_1(\tilde{z}) = \tilde{\Delta}\langle\rho(\epsilon=\tilde{\Delta}\tilde{z}/\pi)\rangle \sim \mathrm{Re}\left[1+\frac{1}{8}\langle\langle(A_B-A_F)^2\rangle\rangle_{\mathrm{kin}}\frac{d}{d\tilde{z}}\tilde{z}^2\frac{d}{d\tilde{z}}\right]\rho_1^{(0)}(\tilde{z}), \tag{75}$$

where

$$\langle\langle(A_B-A_F)^2\rangle\rangle_{\mathrm{kin}} = \langle(A_B-A_F)^2\rangle_{\mathrm{kin}} - \langle A_B-A_F\rangle_{\mathrm{kin}}^2 \sim \frac{a_d}{g^2}. \tag{76}$$

This is obtained by expanding $\tilde{Q}$ in powers of P and using the contraction (A6) with $\epsilon=0$. a_d is momentum summation and is given by Eq. (39). Thus we obtain Eq. (38).

5. AA's domain $(1\ll z)$

In the limit $1\ll z$, the asymptotic form of the Bessel function (60) is used to write

$$\langle\rho(\epsilon)\rangle_2 \sim \frac{1}{\Delta}\mathrm{Re}\frac{d}{dz}\int_z^\infty dt\left[-\frac{t-z}{4t^3}\langle e^{iz(A_B+A_F)}\rangle_{\mathrm{kin}} - i\frac{t-z}{t}\langle e^{2it+iz(A_B-A_F)}\rangle_{\mathrm{kin}}\right] \sim \frac{1}{\Delta}\mathrm{Re}\left[\frac{1}{8z^2}\mathcal{D}(z,1,1)-\frac{1}{2z}e^{2iz}\mathcal{D}(z,1,-1)\right], \tag{77}$$

where

$$\mathcal{D}(z,\lambda_B,\lambda_F) = \int \mathcal{D}\tilde{Q}e^{-F(z,\lambda_B,\lambda_F)},$$

$$F(z,\lambda_B,\lambda_F) = F_{\mathrm{kin}} + \frac{iz\lambda_F}{2V}\int \mathrm{str}\, k_F\Sigma_z(\tilde{Q}-\Sigma_z) + \frac{iz\lambda_B}{2V}\int \mathrm{str}\, k_B\Sigma_z(\tilde{Q}-\Sigma_z). \tag{78}$$

$F(z,1,1)=F_1$ does not break supersymmetry, which means it does not include the supermatrix $k=\mathrm{diag}(1,-1)$. As a result we obtain $\mathcal{D}(z,1,1)=1$. On the other hand, $F(z,1,-1)$ breaks supersymmetry and the function $\mathcal{D}(z,1,-1)$ is not normalized to unity. It is calculated as $\mathcal{D}(z,1,-1)\sim\mathcal{D}(z)$, where the spectral determinant $\mathcal{D}(z)$ is given by Eq. (42). We used the approximation of keeping second order in P for $F(z,1,-1)$. We refer to Appendix A for details (see also the following paragraph).

Equation (77) is rewritten in terms of the energy variable scaled by the renormalized mean level spacing $\tilde{z}=\pi\epsilon/\tilde{\Delta}$. We use the formula (74) and the difference between Δ and $\tilde{\Delta}$ is expressed by the diffusion propagator $\Pi(q,\epsilon)$. It represents the self-interacting diffusion bubble and should be canceled out. Actually we have contributions from the function $\mathcal{D}(z,1,-1)$ by keeping higher-order terms in P. We find

$$\mathcal{D}(z,1,-1) \sim \int \mathcal{D}\tilde{Q}e^{-F^{(2)}(z,1,-1)}[1-F^{(4)}(z,1,-1)] = \mathcal{D}(z)\left[1+iz\,\mathrm{Re}\left(\sum_{q\neq 0}\Pi(q,\epsilon)\right)^2\right], \tag{79}$$

where $F^{(n)}$ denotes the nth order part in the expansion. The second term cancels with a contribution coming from the transformation (74). Noting $\mathcal{D}(z,g)=\mathcal{D}(\tilde{z},\tilde{g})$, where $g=\pi E_c/\Delta$ and $\tilde{g}=\pi E_c/\tilde{\Delta}$, we finally obtain the result Eq. (41) for $1\ll z$.

C. Comparison with the KM method

The obtained result (38) for the KM domain differs slightly from Eq. (21) in Ref. [14] by the presence of momentum integration of the propagator $\Sigma_{q\neq 0}\Pi(q,0)$. As we can understand from Eq. (37), the difference comes from the introduction of the renormalized mean level spacing [Eq. (73) coincides with Eq. (21) in Ref. [14]]. It is expressed as a self-interacting diffusion diagram [it can be understood by noting the coordinate representation $\Sigma_q\Pi(q)=\Pi(x,x)$] and is renormalized to the mean level spacing. In order to make this difference clear, we repeat the calculation using KM's method considered in Ref. [14]. In this method, the nonzero modes are integrated out while keeping the zero mode variables. It allows us to obtain the renormalized effective zero-mode action and is useful to understand how we can introduce the renormalized mean level spacing.

We start from the functional for the DOS with the source term

$$F = \frac{\pi D}{4\Delta V}\int \mathrm{str}[\nabla Q(x)]^2 + \frac{i\pi\epsilon}{2\Delta V}\int \mathrm{str}\, Q(x)\Sigma_z + \frac{i\pi J}{2\Delta V}\int \mathrm{str}\, kQ(x)\Sigma_z. \tag{80}$$

The Q-matrix parametrization (31) is substituted and the nonzero modes $\tilde{Q}$ are expanded in P as Eq. (28). In our approximation, keeping second order in $1/g$, the expansion is performed up to fourth order in P. The functional F consists of four parts:

$$F = F_0 + \tilde{F} + F_I + F_J. \tag{81}$$

F_0 is the zero-mode part $F_0=F[Q=T\Sigma_z\overline{T}]$, $\tilde{F}$ the nonzero-mode part $\tilde{F}=F[\tilde{Q}]$, F_I the mixing part, and F_J the source term. They are expanded in P as

$$\tilde{F} = \tilde{F}^{(2)} + \tilde{F}^{(4)} + \cdots,$$

$$F_I = F_I^{(2)} + F_I^{(3)} + F_I^{(4)} + \cdots,$$

$$F_J = F_J^{(0)} + F_J^{(2)} + F_J^{(3)} + F_J^{(4)} + \cdots, \tag{82}$$

where $F^{(n)}$ denotes the nth order part in P.

The effective functional is obtained by integrating the nonzero modes. We define F_{eff} as

$$e^{-F_{\rm eff}} = \int \mathcal{D}\tilde{Q} e^{-F} = e^{-F_0}\langle e^{-\tilde{F}^{(4)}+\cdots-F_I-F_J}\rangle_{\tilde{F}^{(2)}}, \quad (83)$$

where the average is performed with respect to $\tilde{F}^{(2)}$. We use the contraction rules derived in Appendix A. Up to second order in the cumulant expansion,

$$\begin{aligned}F_{\rm eff} &\sim F_0 + F_J^{(0)} + \langle F_I^{(4)}\rangle + \langle F_J^{(4)}\rangle - \frac{1}{2}\langle\langle F_I^{(2)2}\rangle\rangle - \langle\langle F_I^{(2)}F_J^{(2)}\rangle\rangle \\ &= \frac{i\pi\epsilon}{2\Delta}\left[1+\frac{1}{2}\left(\sum_{q\neq 0}\Pi(q,\epsilon)\right)^2\right]\mathrm{str}\, Q\Sigma_z \\ &\quad + \frac{\pi^2\epsilon^2}{8\Delta^2}\left(\sum_{q\neq 0}\Pi^2(q,\epsilon)\right)(\mathrm{str}\, Q\Sigma_z)^2 \\ &\quad + \frac{i\pi J}{2\Delta}\left[1+\frac{1}{2}\left(\sum_{q\neq 0}\Pi(q,\epsilon)\right)^2\right]\mathrm{str}\, k\Sigma_z Q \\ &\quad + \frac{\pi^2\epsilon J}{4\Delta^2}\left(\sum_{q\neq 0}\Pi^2(q,\epsilon)\right)\mathrm{str}\, Q\Sigma_z \mathrm{str}\, k\Sigma_z Q. \end{aligned} \quad (84)$$

Since momentum summations potentially involve divergence, this expansion is somewhat cumbersome. This can be clearly seen by considering the KM domain $z \ll g$. Then the energy ϵ in the propagator is neglected in our approximation $\Pi(q,\epsilon)\sim\Pi(q,0)$ and the effective functional can be written as

$$\begin{aligned}F_{\rm eff} &\sim \frac{i\pi\epsilon}{2\Delta}\left[1+\frac{a_d^{(1)2}}{8g^2}\right]\mathrm{str}\, Q\Sigma_z + \frac{\pi^2\epsilon^2}{8\Delta^2}\frac{a_d}{4g^2}(\mathrm{str}\, Q\Sigma_z)^2 \\ &\quad + \frac{i\pi J}{2\Delta}\left[1+\frac{a_d^{(1)2}}{8g^2}\right]\mathrm{str}\, k\Sigma_z Q + \frac{\pi^2\epsilon J}{4\Delta^2}\frac{a_d}{4g^2}\mathrm{str}\, Q\Sigma_z \mathrm{str}\, k\Sigma_z Q,\end{aligned} \quad (85)$$

where a_d is given by Eq. (39) and

$$a_d^{(1)} = \frac{1}{\pi^2}\sum_{n_i\geq 0, n^2\neq 0}\frac{1}{n^2}. \quad (86)$$

This summation is divergent at $d\geq 2$ and we need some regularization. Fortunately, and as it should be, the quantity $a_d^{(1)}$ can be renormalized to the mean level spacing by defining the renormalized spacing

$$\frac{1}{\tilde{\Delta}} = \frac{1}{\Delta}\left[1+\frac{a_d^{(1)2}}{8g^2}+O(1/g^3)\right]. \quad (87)$$

This is nothing but the expression (37) at the KM domain. $a_d^{(1)}$ corrections come from the average $\langle Q(x)\rangle$. On the other hand the second and fourth terms in Eq. (85) come from the contraction $\langle\langle Q(x)Q(y)\rangle\rangle$ and cannot be renormalized to Δ. They give the corrections obtained in Eq. (38).

Thus the KM method makes the problem of the renormalization transparent. The idea of integrating out fast variables matches the philosophy of the renormalization. Nevertheless, we did not use this method for the reason that it is not convenient for calculations in the AA domain $z\gg 1$. Integrations of zero-mode variables naturally bring contributions from nontrivial saddle points, which is an important idea for nonperturbative calculations.

D. Two-level correlation function

Now we turn to the calculation of the TLCF (26). Q is an 8×8 supermatrix and the explicit parametrization is different from the previous case.

For the standard perturbative calculation, we use the expansion Eq. (28). The explicit parameterization of the P matrix is given by

$$P = \begin{pmatrix} 0 & t \\ \bar{t} & 0\end{pmatrix}, \quad t = \begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2\end{pmatrix},$$

$$t_1 = \begin{pmatrix} a_1 & \sigma_1 \\ \rho_1 & ib_1\end{pmatrix}, \quad t_2 = \begin{pmatrix} a_2 & \sigma_2 \\ \rho_2 & ib_2\end{pmatrix},$$

$$t_{12} = \begin{pmatrix} c & i\eta \\ \xi^* & id\end{pmatrix}, \quad t_{21} = \begin{pmatrix} c^* & \xi \\ i\eta^* & id^*\end{pmatrix}. \quad (88)$$

$a_{1,2}$, $b_{1,2}$ are real variables, c, d complex variables, and the greek symbols denote Grassmann variables. As the explicit parametrization implies, $t_{1,2}$ represent the "chiral" part and $t_{12,21}$ the "unitary" part. Starting from the expression (27), we have

$$\begin{aligned}W(\epsilon_1,\epsilon_2) = \frac{1}{\Delta^2}\int \mathcal{D}Q e^{-F_2^{(0)}(z_1,z_1,z_2,z_2)+\cdots} &\times\left[1-\frac{1}{2V}\int_x \mathrm{str}\, k\Lambda_1 P^2(x)+\cdots\right] \\ &\times\left[1-\frac{1}{2V}\int_y \mathrm{str}\, k\Lambda_2 P^2(y)+\cdots\right].\end{aligned} \quad (89)$$

$F_2^{(2)}(z_1,z_1,z_2,z_2)$ given by Eq. (A11) is second order in P and is the base of the perturbative expansion. The contraction rule given by Eq. (A14) is used to evaluate the above expression. The leading order contribution to the connected part comes from the contraction

$$\langle \mathrm{str}\, k\Lambda_1 P^2(x)\mathrm{str}\, k\Lambda_2 P^2(y)\rangle = 4\Pi^2(x-y,(\epsilon_1+\epsilon_2)/2). \quad (90)$$

Thus we obtain the result (36) which is valid at $g\gg 1$ and $z_{1,2}=\pi\epsilon_{1,2}/\Delta\gg 1$.

The ergodic limit $g\to\infty$ was considered in Ref. [23]. The Q matrix is parametrized as

$$Q = T\Sigma_z\overline{T}, \quad T = T_{\rm ch}T_{\rm u}. \quad (91)$$

$T_{\rm ch}$ is the chiral part

$$T_{\rm ch} = U_{\rm ch}T_{\rm ch}^{(0)}\overline{U_{\rm ch}},$$

$$T_{\rm ch}^{(0)}=\begin{pmatrix}\cos\frac{\hat\theta_1}{2} & 0 & -i\sin\frac{\hat\theta_1}{2} & 0\\ 0 & \cos\frac{\hat\theta_2}{2} & 0 & -i\sin\frac{\hat\theta_2}{2}\\ -i\sin\frac{\hat\theta_1}{2} & 0 & \cos\frac{\hat\theta_1}{2} & 0\\ 0 & -i\sin\frac{\hat\theta_2}{2} & 0 & \cos\frac{\hat\theta_2}{2}\end{pmatrix},$$

$$\hat\theta=\begin{pmatrix}\hat\theta_1 & 0\\ 0 & \hat\theta_2\end{pmatrix}=\begin{pmatrix}\theta_{1F} & 0 & 0 & 0\\ 0 & i\theta_{1B} & 0 & 0\\ 0 & 0 & \theta_{2F} & 0\\ 0 & 0 & 0 & i\theta_{2B}\end{pmatrix},$$

$$U_{\rm ch}=\begin{pmatrix}u_{\rm ch1} & 0 & 0 & 0\\ 0 & u_{\rm ch2} & 0 & 0\\ 0 & 0 & u_{\rm ch1} & 0\\ 0 & 0 & 0 & u_{\rm ch2}\end{pmatrix},\quad u_{\rm ch1,2}=\exp\begin{pmatrix}0 & \sigma_{1,2}\\ \rho_{1,2} & 0\end{pmatrix}, \tag{92}$$

and $T_{\rm u}$ the unitary part

$$T_{\rm u}=U_{\rm u}T_{\rm u}^{(0)}\overline{U_{\rm u}},$$

$$T_{\rm u}^{(0)}=\begin{pmatrix}\cos\frac{\hat\Omega}{2} & 0 & 0 & -ie^{i\hat\varphi}\sin\frac{\hat\Omega}{2}\\ 0 & \cos\frac{\hat\Omega}{2} & -ie^{-i\hat\varphi}\sin\frac{\hat\Omega}{2} & 0\\ 0 & -ie^{i\hat\varphi}\sin\frac{\hat\Omega}{2} & \cos\frac{\hat\Omega}{2} & 0\\ -ie^{-i\hat\varphi}\sin\frac{\hat\Omega}{2} & 0 & 0 & \cos\frac{\hat\Omega}{2}\end{pmatrix},$$

$$\hat\Omega=\begin{pmatrix}\Omega_F & 0\\ 0 & i\Omega_B\end{pmatrix},\quad \hat\varphi=\begin{pmatrix}\varphi_F & 0\\ 0 & \varphi_B\end{pmatrix},$$

$$U_{\rm u}=\begin{pmatrix}u_{\rm u1} & 0 & 0 & 0\\ 0 & u_{\rm u2} & 0 & 0\\ 0 & 0 & u_{\rm u1} & 0\\ 0 & 0 & 0 & u_{\rm u2}\end{pmatrix},$$

$$u_{\rm u1}=\exp\begin{pmatrix}0 & \xi\\ -\xi^* & 0\end{pmatrix},\quad u_{\rm u2}=\exp\begin{pmatrix}0 & i\eta\\ -i\eta^* & 0\end{pmatrix}. \tag{93}$$

σ, ρ, ξ, and η are Grassmann variables. The integration ranges of the real variables θ, Ω, and φ are chosen properly according to the compact or noncompact parametrization [23]. The measure is given by

$$\begin{aligned}\mathcal{D}Q&=d\theta_{1B}d\theta_{1F}d\sigma_1d\rho_1\frac{1}{4\pi}\frac{\cosh\theta_{1B}\cos\theta_{1F}-1}{(\cosh\theta_{1B}-\cos\theta_{1F})^2}d\theta_{2B}\\&\times d\theta_{2F}d\sigma_2d\rho_2\frac{1}{4\pi}\frac{\cosh\theta_{2B}\cos\theta_{2F}-1}{(\cosh\theta_{2B}-\cos\theta_{2F})^2}\\&\times d\Omega_Bd\Omega_F\frac{d\varphi_B}{2\pi}\frac{d\varphi_F}{2\pi}d\xi\, d\xi^*\, d\eta^*\\&\times d\eta\frac{\sinh\Omega_B\sin\Omega_F}{(\cosh\Omega_B-\cos\Omega_F)^2}\frac{4\cosh\Omega_B\cos\Omega_F}{(\cosh\Omega_B+\cos\Omega_F)^2}.\end{aligned} \tag{94}$$

Using this parametrization, after a laborious calculation, we can obtain Eq. (33) (see Ref. [23] for the details).

The nonperturbative calculation using the parametrization (31) can be done in the same way as that of the DOS. First we integrate the zero-mode variables. The details are presented in Appendix B, and we find for W

$$W=W_1+W_2,$$

$$\begin{aligned}W_1&\sim\frac{1}{\Delta^2}\Bigg\langle\Bigg\{\left[1+\frac{1}{2}(A_{B1}-A_{F1})\right]e^{iz_1(A_{-1}+A_{+1})}\\&+\frac{\pi}{2}\frac{\partial}{\partial z_1}\int_{z_1}^{\infty}dt_1(t_1-z_1)[J_0(t_1+z_1A_{B1})J_0(t_1-z_1A_{F1})\\&-J_1(t_1+z_1A_{B1})J_1(t_1-z_1A_{F1})]\Bigg\}\Bigg\{\left[1+\frac{1}{2}(A_{B2}\right.\\&\left.-A_{F2})\right]e^{iz_2(A_{B2}+A_{F2})}+\frac{\pi}{2}\frac{\partial}{\partial z_2}\int_{z_2}^{\infty}dt_2(t_2-z_2)\\&\times[J_0(t_2+z_2A_{B2})J_0(t_2-z_2A_{F2})\\&-J_1(t_2+z_2A_{B2})J_1(t_2-z_2A_{F2})]\Bigg\}\Bigg\rangle_{\rm kin},\end{aligned}$$

$$W_2\sim\frac{1}{\Delta^2}\int ds_1ds_2\frac{4s_1s_2}{(s_1^2-s_2^2)^2}I(z,s),$$

$$\begin{aligned}I(z,s)&=\frac{\pi^2}{4}\Bigg\langle z_1z_2(s_1-s_2+C_1)(s_1+s_2-D_1)J_0\Bigg(z_1s_1\\&+z_1\frac{C_1-D_1}{2}\Bigg)J_0\Bigg(z_1s_2-z_1\frac{C_1+D_1}{2}\Bigg)(s_1-s_2+C_2)(s_1\\&+s_2-D_2)J_0\Bigg(z_2s_1+z_2\frac{C_2-D_2}{2}\Bigg)J_0\Bigg(z_2s_2\\&-z_2\frac{C_2+D_2}{2}\Bigg)\Bigg\rangle_{\rm kin},\end{aligned} \tag{95}$$

where $s_1=\cosh\Omega_B$, $s_2=\cos\Omega_F$, and

$$A_{B1,2}=-\frac{1}{2V}\int\,{\rm str}\,k_B\Lambda_{1,2}\Sigma_z(\tilde Q-\Sigma_z),$$

$$A_{F1,2}=-\frac{1}{2V}\int\,{\rm str}\,k_F\Lambda_{1,2}\Sigma_z(\tilde Q-\Sigma_z),$$

$$C_{1,2}=\frac{1}{2}[s_1A_B+s_2A_F\pm(A_{B\Lambda}+A_{F\Lambda})],$$

$$D_{1,2}=\frac{1}{2}[-s_1A_B+s_2A_F\pm(-A_{B\Lambda}+A_{F\Lambda})],$$

$$A_{F,B}=A_{F,B1}+A_{F,B2},$$

$$A_{F,B\Lambda}=A_{F,B1}-A_{F,B2}. \tag{96}$$

We neglected odd terms in P as before. W_1 includes the perturbative contributions (36) and W_2 includes the ergodic result (33).

In the KM domain $z_{1,2}\ll g$, the expansion in zA can be used. The integrations of the nonzero modes are calculated up to second order in $1/g$. Introducing the renormalized mean level spacing we have

$$\begin{aligned}\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle\sim\frac{1}{\tilde{\Delta}^2}\Bigg[1+\frac{a_d}{4g^2}+\frac{a_d}{2g^2}\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}\right)\\+\frac{a_d}{8g^2}\left(z_1^2\frac{\partial^2}{\partial z_1}+2z_1z_2\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}+z_2^2\frac{\partial^2}{\partial z_2}\right)\Bigg]\\\times[\rho_1^{(0)}(z_1)\rho_1^{(0)}(z_2)-K^2(z_1,z_2)],\end{aligned} \tag{97}$$

where $z_{1,2}=\pi\epsilon_{1,2}/\tilde{\Delta}$, and $\tilde{\Delta}$ is given by Eq. (87). Subtracting the disconnected part, we derive Eq. (40).

The AA domain $1\ll z_{1,2}$, $1\ll g$ is considered using the asymptotic form of the Bessel function (60). The details are presented in Appendix B. From W_1 we obtain the first part

$$\begin{aligned}\langle\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle\rangle_1\sim\frac{1}{\Delta^2}\Bigg[\frac{1}{2}\mathrm{Re}\sum_{q^2\neq0}(\Pi_+^2+\Pi_-^2)\\+\frac{\sin 2z_1}{2z_1}\mathcal{D}_1\mathrm{Im}\sum_{q^2\neq0}(\Pi_++\Pi_-)\\+\frac{\sin 2z_2}{2z_2}\mathcal{D}_2\mathrm{Im}\sum_{q^2\neq0}(\Pi_+-\Pi_-)\\+\frac{\cos 2(z_1+z_2)}{8z_1z_2}\mathcal{D}_1\mathcal{D}_2(\mathcal{D}_+^2+\mathcal{D}_-^{-2}-1)\\+\frac{\cos 2(z_1-z_2)}{8z_1z_2}\mathcal{D}_1\mathcal{D}_2(\mathcal{D}_-^2\mathcal{D}_+^{-2}-1)\Bigg],\end{aligned} \tag{98}$$

where $\mathcal{D}_{1,2}=\mathcal{D}(z_{1,2})$, $\mathcal{D}_\pm=\mathcal{D}((z_1\pm z_2)/2)$, and $\Pi_\pm=\Pi(q,(\epsilon_1+\epsilon_2)/2)$. The first term represents the purely perturbative contribution. The second connected part W_2 is calculated in the same way. We obtain

$$\begin{aligned}\langle\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle\rangle_2\sim-\frac{1}{\Delta^2}\mathrm{Re}\Bigg\{\frac{1-e^{2i(z_1-z_2)}\mathcal{D}_1\mathcal{D}_2\mathcal{D}_-^2\mathcal{D}_+^{-2}}{2(z_1-z_2)^2}\\+\frac{(1+e^{2i(z_1+z_2)}\mathcal{D}_1\mathcal{D}_2\mathcal{D}_+^2\mathcal{D}_-^{-2})}{2(z_1+z_2)^2}\\+\frac{i(e^{2iz_1}\mathcal{D}_1-e^{2iz_2}\mathcal{D}_2)}{z_1^2-z_2^2}\Bigg\}.\end{aligned} \tag{99}$$

The derived expressions are written in terms of the unrenormalized quantity Δ and we must carry out rescaling in terms of $\tilde{\Delta}$. Additional contributions coming from the rescaling should cancel out with terms we did not show explicitly here. This situation is the same as the DOS case and we finally arrive at Eq. (43).

IV. THE DOUBLE-TRACE TERM AND THE DOS RENORMALIZATION

In the previous section, we neglected the second term in Eqs. (16) and (19). This double-trace term includes nonzero modes only and changes the perturbative result. It appears only in systems with chiral symmetry and we therefore concentrate on the DOS.

At second order in P, we have instead of Eq. (51)

$$\begin{aligned}F_1^{(0)}=\frac{\pi D}{\Delta V}\int\mathrm{str}(\nabla P)^2+\frac{\pi D_1}{4\Delta V}\int(\mathrm{str}\ \nabla P\Sigma_x)^2\\-\frac{i\pi\epsilon}{\Delta V}\int\mathrm{str}\,P^2.\end{aligned} \tag{100}$$

The presence of the second term modifies the contraction rule as Eq. (A9). In this case perturbation theory is formulated by expansions in Π, Eq. (29), and

$$\Pi_2(q,\epsilon)=\frac{\pi D_1q^2}{\Delta}\Pi^2(q,\epsilon). \tag{101}$$

The corresponding expansion parameters are $1/g\propto1/D$ and $g_1/g^2\propto D_1/D^2$.

The perturbative expansion gives the DOS

$$\begin{aligned}\langle(\epsilon)\rangle=\frac{1}{\Delta}\mathrm{Re}\Bigg[1+\sum_q\Pi_2(q,\epsilon)+\frac{1}{2}\left(\sum_q\Pi(q,\epsilon)\right)^2\\+\frac{1}{2}\left(\sum_q\Pi_2(q,\epsilon)\right)^2+\cdots\Bigg].\end{aligned} \tag{102}$$

The new propagator Π_2 contributes to the DOS at one-loop order. The renormalized mean level spacing is defined as the inverse of Eq. (102) excluding zero-mode contributions. Actually this result was derived by Gade using the renormalization group method [17]. In our model (16), following the calculation in Ref. [6], we can obtain the same renormalization group equations at one-loop order as

$$\beta_b = -\frac{db}{d\mu} = \epsilon b, \quad \beta_c = -\frac{dc}{d\mu} = \epsilon c + c^2, \quad \zeta = \frac{b^2}{c}, \tag{103}$$

where $b \sim 1/g$ and $c \sim 1/g$. We used ϵ expansion, $\epsilon = d-2$, and μ is the renormalization scale. $\beta_{b,c}$ are the beta functions for b and c. ζ is the zeta function for the wave function renormalization and corresponds to the result in Eq. (102). These equations imply a divergence of the DOS and delocalization of eigenstates in two dimensions. Thus the presence of the double-trace term changes the behavior of the DOS significantly. We note that the renormalization procedure produces the double-trace term even if we start the analysis from a model without that term. The quantum effect in two dimensions increases the coupling constant c.

We consider the scaled DOS $\rho_1(z)$ to examine how the double-trace term contributes to the result. The perturbative result (102) is renormalized to the mean level spacing. In a similar way as the calculation in the previous section we find in the KM domain $z \ll g$

$$\rho_1(z) \sim \left\{1 + \left[\frac{a_d}{8g^2} + \frac{a_d}{16}\left(\frac{g_1}{g^2}\right)^2\right]\left(2z\frac{d}{dz} + z^2\frac{d^2}{dz^2}\right)\right\}\rho_1^{(0)}(z). \tag{104}$$

In the AA domain $1 \ll z$, the spectral determinant is modified as Eq. (A10). Subtracting the renormalization effect, we obtain

$$\mathcal{D}(z) \sim \prod_{q \geqslant 0, q^2 \neq 0} \frac{(Dq^2)^2}{(Dq^2)^2 + \epsilon^2}\left[1 - 8z^2 \sum_{q \geqslant 0, q^2 \neq 0} |\Pi_2(q,\epsilon)|^2\right], \tag{105}$$

which is consistent with Eq. (104).

Finally we mention the Jacobian contribution in Eq. (66). It includes a term second order in P and changes the contraction rules. Since this term is similar to the last term in Eq. (100) it can be easily incorporated into the contraction rules by the replacement

$$\Pi_2(q,\epsilon) \to \Pi_2(q,\epsilon) - \Pi^2(q,\epsilon). \tag{106}$$

Thus this Jacobian contribution is always subleading compared to the propagator Π_2. We also note that this contributes only to $\tilde{\Delta}$ and not to the scaled DOS $\rho_1(z)$ in our approximation.

V. DISCUSSION AND CONCLUSIONS

We have studied disordered systems with chiral unitary symmetry. Using a chiral symmetric random matrix model we derived the nonlinear σ models (16) and (19). We demonstrated that they are equivalent to related chiral symmetric models. Using the σ models, we calculated the level correlation functions. We exploited the nonperturbative methods developed by Kravtsov and Mirlin and Andreev and Altshuler for the traditional classes.

The equivalence of the models shows the universality of disordered systems. Our derived σ models are applicable to models treated in Refs. [17,18,20]. The double-trace term was not derived in Ref. [20]. This is because the massive mode integration was not considered carefully.

For the calculation of the DOS and TLCF, we stressed the need for the renormalization of the mean level spacing. This renormalization is absent in traditional nonchiral systems. After separating the renormalization effect, we found the results (38) and (40) in the KM domain and (41) and (43) in the AA domain. It is interesting to note that the results in the AA domain are expressed using the spectral determinant. It contributes to oscillating terms only, in a similar way as for the traditional classes. Thus we conclude that the singularity of the form factor at the Heisenberg time is washed out due to finite-g effects [9].

Our formulation of the perturbative and nonperturbative calculations can be useful not only for the level correlation functions but also for the conductance and other quantities. In the present work we concentrated on the level correlation functions. In Ref. [20], the same quantities were calculated perturbatively. The different result obtained there is due to another parametrization of the Q matrix. Additional contributions coming from the integration measure would give the correct result. In Ref. [14], the DOS in the KM domain was calculated from the model derived in Ref. [20]. The result was scaled in terms of the bare mean level spacing Δ, and the renormalized mean level spacing $\tilde{\Delta}$ was not introduced. This leads us to a different conclusion on level statistics as we mention below.

Let us discuss the importance of introducing the renormalized mean level spacing. There are numerous works on the behavior of the DOS at the origin $\epsilon = 0$. The main question is whether the DOS diverges or not, and analytically it has been considered using perturbation theory at weak disorder $g \gg 1$. On the other hand, chiral RMT, which corresponds to the model at $g = \infty$, predicts the vanishing of the DOS at the origin $z = 0$. This is not a contradiction and indicates the importance of scaling. The macroscopic behavior is determined by the nonzero modes and a divergence of $1/\tilde{\Delta}$ was reported in Ref. [17]. The zero mode has nothing to do with this behavior and determines the universal behavior at the microscopic scale. It can be seen by scaling the energy variable ϵ in terms of the mean level spacing.

Generally speaking, the behavior at the macroscopic scale depends on the model. From a field theoretical point of view, the divergence can be renormalized to the mean level spacing and a definite conclusion as to whether it is a real divergence or not can be obtained by referring to other approaches such as numerical simulations. Our result relies on perturbation theory and the divergence may be cut off somewhere before the origin. This crossover to the universal microscopic domain is highly nonperturbative. Since a high resolution is required, it may be hard to see such a crossover numerically.

From the viewpoint of level statistics, the DOS must be scaled (renormalized) to unity at all energies to find the universality. This unfolding procedure cannot be applied to the present chiral case because the DOS itself has universal fine structure (oscillations due to level repulsion) at the origin. For this reason, we use Δ, the (inverse) DOS at $z = \infty$ $(\epsilon = 0)$, for scaling in the ergodic regime. It is modified by finite-g

effects and we use $\tilde{\Delta}$ to see the microscopic domain closely.

Thus using the renormalized mean level spacing, we can separate problems at both scales. The effects of nonzero modes (finite-g effect) cannot be scaled out completely in the microscopic domain and deviations from the universal behavior are obtained as we have shown in the present work. Such an example can be found in Ref. [16]. The generalized random matrix model was used there and it was found that the quantity $\tilde{\Delta}$ is different from our result. However, after scaling in terms of the nonuniversal quantity $\tilde{\Delta}$, we can find complete agreement up to finite-g corrections. This demonstration of "universal deviation" justifies the introduction of $\tilde{\Delta}$.

The double-trace term contribution is small at the classical level because the coupling constant is small compared with that in the diffusion propagator. However, quantum effects affect this coupling and the contribution becomes important in some cases. It significantly affects the DOS renormalization and a diverging DOS was found in Ref. [17]. Concerning level statistics, this term modifies the spectral determinant as Eq. (A10).

Our calculation is only for the chiral unitary class. The other chiral classes, chiral orthogonal and symplectic, can be calculated in the same way. The problem is that the proper parametrization of the zero mode has not been found. However the KM domain can be considered without knowing the zero-mode parametrization as was done in Ref. [14]. The obtained result is valid only at first order in $1/g$. Repeating the same calculation up to the next order and introducing the renormalized mean level spacing, we found the same form as Eq. (38). The coefficient of the second term in Eq. (38) is changed but with the same sign for all the classes. This result also holds for the TLCF (40). We thus obtain the same conclusion as KM, namely, the weakening of level repulsion [this can be seen, e.g., in Eq. (48)]. The authors in Ref. [14] drew a different conclusion by looking at the first order correction to the mean level spacing. It is renormalized to the mean level spacing and should be applied to the DOS behavior and not to level repulsion.

As an interesting application, we mention a related work in Ref. [29]. For traditional nonchiral systems, the authors in Ref. [13] pointed out that the AA result is related to the Calogero-Sutherland model at finite temperature. It is shown in Ref. [30] that this model is equal to the generalized random matrix model proposed in Ref. [31]. In this problem the nonlinear σ model is modified due to power-law correlations of random matrices [15] and the diffusion propagator and spectral determinant are modified. As a result agreement with the result in Ref. [31] was found and a conjecture to more general cases was made. We expect this holds also for chiral systems and the result is presented in Ref. [16].

Another future problem is the wave function statistics. For traditional classes, it was considered in Ref. [32] using the KM method. It will be interesting to see how this result is modified in the chiral symmetric case.

ACKNOWLEDGMENTS

The author is grateful to A. M. García-García, S. Iida, and G. Schwiete for useful discussions and reading the manuscript. The financial support by the SFB/Transregio 12 is acknowledged.

APPENDIX A: CONTRACTION RULES

1. Calculation for F_1

Consider the functional

$$F_1^{(2)} = \frac{\pi D}{\Delta V}\int \mathrm{str}(\nabla P)^2 - \frac{iz}{V}\lambda_F\int \mathrm{str}\, k_F P^2 - \frac{iz}{V}\lambda_B\int \mathrm{str}\, k_B P^2, \tag{A1}$$

where $k_{F,B}=(1\pm k)/2$ and $z=\pi\epsilon/\Delta$. The P matrix is a 4×4 supermatrix including nonzero modes and is given by Eq. (50). Since this functional breaks supersymmetry for $\lambda_F \neq \lambda_B$, the function $\mathcal{D}_1(z,\lambda_B,\lambda_F)=\int \mathcal{D}\tilde{Q}\exp(-F_1^{(2)})$ is not normalized to unity. We calculate this function and derive the contraction rules.

Using the explicit parametrization (50), we write $F_1^{(2)}$ as

$$F_1^{(2)} = \sum_{q\neq 0}\begin{pmatrix} -\rho & \sigma & a & b\end{pmatrix}(-q)G^{-1}\begin{pmatrix}\sigma\\ \rho\\ a\\ b\end{pmatrix}(q)$$

$$\equiv \sum_{q\neq 0}\overline{\psi}(q)G^{-1}\psi(q),$$

$$G = \mathrm{diag}[\Pi(q,\epsilon\lambda_+),\Pi(q,\epsilon\lambda_+),\Pi(q,\epsilon\lambda_F),\Pi(q,\epsilon\lambda_B)], \tag{A2}$$

where $\lambda_+=(\lambda_B+\lambda_F)/2$ and the diffusion propagator is given by Eq. (29). Then the functional integral is given by

$$\mathcal{D}_1(z,\lambda_B,\lambda_F) = \int \mathcal{D}(\overline{\psi},\psi)e^{-F_1^{(2)}} = \prod_{q^2\neq 0}\left[\frac{\Pi(q,\epsilon\lambda_B)\Pi(q,\epsilon\lambda_F)}{\Pi^2(q,\epsilon\lambda_+)}\right]^{1/2} = \prod_{q\geqslant 0,q^2\neq 0}\frac{(Dq^2-i\epsilon\lambda_+)^2}{(Dq^2-i\epsilon\lambda_B)(Dq^2-i\epsilon\lambda_F)}, \tag{A3}$$

$$\int \mathcal{D}(\overline{\psi},\psi)\psi\overline{\psi}e^{-F_1^{(2)}} = \frac{1}{2}\mathcal{D}_1(z,\lambda_B,\lambda_F)G. \tag{A4}$$

Since $\psi(q)$ and $\psi(-q)$ are not independent of each other, the square root appears in $\mathcal{D}$. We used the periodic boundary condition and $q_i=2\pi n_i/L$, n_i is integer.

Using the result we obtain the contraction rules for the matrix P as

$$\langle \mathrm{str}\, AP(x)BP(y)\rangle = \frac{1}{4}\sum_{i,j=F,B}\Pi\left(x-y,\epsilon\frac{\lambda_i+\lambda_j}{2}\right)(\mathrm{str}\, k_iA\,\mathrm{str}\, k_jB - \mathrm{str}\, k_iA\Sigma_z\mathrm{str}\, k_jB\Sigma_z + \mathrm{str}\, k_iA\Sigma_x\mathrm{str}\, k_jB\Sigma_x - \mathrm{str}\, k_iA\Sigma_y\mathrm{str}\, k_jB\Sigma_y),$$

$$\langle \mathrm{str}\, AP(x)\mathrm{str}\, BP(y)\rangle = \frac{1}{4}\sum_{i,j=F,B}\Pi\left(x-y,\epsilon\frac{\lambda_i+\lambda_j}{2}\right) \times \mathrm{str}(k_iAk_jB - k_iA\Sigma_zk_jB\Sigma_z + k_iA\Sigma_xk_jB\Sigma_x - k_iA\Sigma_yk_jB\Sigma_y), \tag{A5}$$

where A and B are arbitrary supermatrices and $\langle\cdots\rangle = \mathcal{D}_1^{-1}(z,\lambda_B,\lambda_F)\int\mathcal{D}\widetilde{Q}(\cdots)\exp(-F_1^{(2)})$.

We are mainly interested in the cases $(\lambda_B,\lambda_F)=(1,1)$ and $(1,-1)$. The first case (1, 1) corresponds to standard perturbation theory. The free energy does not break supersymmetry and we find $\mathcal{D}(z,1,1)=1$ and

$$\langle \mathrm{str}\, AP(x)BP(y)\rangle = \frac{1}{4}\Pi(x-y,\epsilon)(\mathrm{str}\, A\, \mathrm{str}\, B - \mathrm{str}\, A\Sigma_z\mathrm{str}\, B\Sigma_z + \mathrm{str}\, A\Sigma_x\, \mathrm{str}\, B\Sigma_x - \mathrm{str}\, A\Sigma_y\mathrm{str}\, B\Sigma_y),$$

$$\langle \mathrm{str}\, AP(x)\mathrm{str}\, BP(y)\rangle = \frac{1}{4}\Pi(x-y,\epsilon)\mathrm{str}(AB - A\Sigma_zB\Sigma_z + A\Sigma_xB\Sigma_x - A\Sigma_yB\Sigma_y). \tag{A6}$$

In the case $(1,-1)$, supersymmetry is broken and this is used for calculations in the AA domain $1\ll z$. The function $\mathcal{D}$ is given by

$$\mathcal{D}_1(z,1,-1) = \prod_{q\geqslant 0,q^2\neq 0}\frac{(Dq^2)^2}{(Dq^2)^2+\epsilon^2} = \prod_{n\geqslant 0,n^2\neq 0}\frac{g^2(4\pi^2n^2)^2}{g^2(4\pi^2n^2)^2+z^2}. \tag{A7}$$

2. Effect of the double-trace term

We consider the effect of the double-trace term. The second term of Eq. (100) is included in Eq. (A1). In this case, the matrix G in Eq. (A2) is replaced by

$$G(q) = \mathrm{diag}(\Pi_+,\Pi_+,C\Pi_F,C\Pi_B) - \frac{\pi D_1q^2}{\Delta}\Pi_F\Pi_BC\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&-i\\0&0&-i&-1\end{pmatrix},$$

$$\Pi_{F,B,+} = \frac{\Delta}{2\pi}\frac{1}{Dq^2-i\epsilon\lambda_{F,B,+}},$$

$$C = \frac{1}{1+(\pi D_1q^2/\Delta)(\Pi_F-\Pi_B)}. \tag{A8}$$

As a result, $\mathcal{D}$ and the contraction rules are modified in the following way. The contraction for $(\lambda_B,\lambda_F)=(1,1)$ is expressed as

$$\langle \mathrm{str}\, AP(x)BP(y)\rangle = \frac{1}{4}\Pi(x-y,\epsilon)(\mathrm{str}\, A\, \mathrm{str}\, B - \mathrm{str}\, A\Sigma_z\mathrm{str}\, B\Sigma_z + \mathrm{str}\, A\Sigma_x\mathrm{str}\, B\Sigma_x - \mathrm{str}\, A\Sigma_y\mathrm{str}\, B\Sigma_y) - \frac{1}{2}\Pi_2(x-y,\epsilon)\mathrm{str}\, A\Sigma_xB\Sigma_x,$$

$$\langle \mathrm{str}\, AP(x)\mathrm{str}\, BP(y)\rangle = \frac{1}{4}\Pi(x-y,\epsilon)\mathrm{str}(AB - A\Sigma_zB\Sigma_z + A\Sigma_xB\Sigma_x - A\Sigma_yB\Sigma_y) - \frac{1}{2}\Pi_2(x-y,\epsilon)\mathrm{str}\, A\Sigma_x\mathrm{str}\, B\Sigma_x, \tag{A9}$$

where the propagator Π_2 in momentum space is given by Eq. (101). For $(\lambda_B,\lambda_F)=(1,-1)$, Eq. (A7) is replaced by

$$\mathcal{D}_1(z,1,-1) = \prod_{q\geqslant 0,q^2\neq 0}\frac{(Dq^2)^2}{(Dq^2)^2-i\epsilon D_1q^2+\epsilon^2}. \tag{A10}$$

3. Calculation for F_2

Consider

$$F_2^{(2)}(z_1,z_2,z_3,z_4) = \frac{\pi D}{\Delta V}\int\mathrm{str}(\nabla P)^2 - \frac{iz_1}{V}\int\mathrm{str}\, k_F\Lambda_1P^2 - \frac{iz_2}{V}\int\mathrm{str}\, k_B\Lambda_1P^2 - \frac{iz_3}{V}\int\mathrm{str}\, k_F\Lambda_2P^2 - \frac{iz_4}{V}\int\mathrm{str}\, k_B\Lambda_2P^2, \tag{A11}$$

where $\Lambda_{1,2}=(1\pm\Lambda)/2$ and $z_{1,2,3,4}=\pi\epsilon_{1,2,3,4}/\Delta$. The P matrix is an 8×8 supermatrix and is parametrized as Eq. (88). This case is considered in the same way as the case of $F_1^{(2)}$. We neglect the double-trace term contribution for simplicity. The result is expressed for the functional integral as

$$\mathcal{D}_2(z_1,z_2,z_3,z_4) = \int\mathcal{D}\widetilde{Q}e^{-F_2^{(2)}(z_1,z_2,z_3,z_4)} = \mathcal{D}_{\mathrm{ch}}(z_1,z_2)\mathcal{D}_{\mathrm{ch}}(z_3,z_4)\mathcal{D}_{\mathrm{u}}(z_1,z_2,z_3,z_4),$$

$$\mathcal{D}_{\mathrm{ch}}(z_1,z_2) = \prod_{q\geqslant 0,q^2\neq 0}\frac{[Dq^2-(i/2)(\epsilon_1+\epsilon_2)]^2}{(Dq^2-i\epsilon_1)(Dq^2-i\epsilon_2)},$$

$$\mathcal{D}_{\mathrm{u}}(z_1,z_2,z_3,z_4) = \prod_{q\geqslant 0,q^2\neq 0}\frac{Dq^2-(i/2)(\epsilon_1+\epsilon_4)}{Dq^2-(i/2)(\epsilon_1+\epsilon_3)}\frac{Dq^2-(i/2)(\epsilon_2+\epsilon_3)}{Dq^2-(i/2)(\epsilon_2+\epsilon_4)}. \tag{A12}$$

For example, $\mathcal{D}_2(-z_1,z_1,z_2,z_2)=\mathcal{D}_1(z_1)$, $D_2(z_1,z_1,-z_2,z_2)=\mathcal{D}_1(z_2)$, and $\mathcal{D}_2(-z_1,z_1,-z_2,z_2)=\mathcal{D}_1(z_1)\mathcal{D}_1(z_2)\mathcal{D}_1^2[(z_1+z_2)/2]\mathcal{D}_1^{-2}[(z_1-z_2)/2]$. The contraction rule is given by

$$\langle \mathrm{str}\, AP(x)BP(y)\rangle = \frac{1}{4}\sum_{\alpha,\beta=F,B}\sum_{i,j=1,2}\Pi_{i\alpha j\beta}(x-y) \times (\mathrm{str}\, k_\alpha\Lambda_i A\, \mathrm{str}\, k_\beta\Lambda_j B - \mathrm{str}\, k_\alpha\Lambda_i\Sigma_z A\, \mathrm{str}\, k_\beta\Lambda_j\Sigma_z B + \mathrm{str}\, k_\alpha\Lambda_i\Sigma_x A\, \mathrm{str}\, k_\beta\Lambda_j\Sigma_x B - \mathrm{str}\, k_\alpha\Lambda_i\Sigma_y A\, \mathrm{str}\, k_\beta\Lambda_j\Sigma_y B),$$

$$\langle \mathrm{str}\, AP(x)\mathrm{str}\, BP(y)\rangle = \frac{1}{4}\sum_{\alpha,\beta=F,B}\sum_{i,j=1,2}\Pi_{i\alpha j\beta}(x-y) \times \mathrm{str}(k_\alpha\Lambda_i A k_\beta\Lambda_j B - k_\alpha\Lambda_i\Sigma_z A k_\beta\Lambda_j\Sigma_z B + k_\alpha\Lambda_i\Sigma_x A k_\beta\Lambda_j\Sigma_x B - k_\alpha\Lambda_i\Sigma_y A k_\beta\Lambda_j\Sigma_y B), \tag{A13}$$

where $\Pi_{i\alpha j\beta}(x)=\Pi(x,(\lambda_{i\alpha}+\lambda_{j\beta})/2)$, and $\lambda_{1F}=\epsilon_1$, $\lambda_{1B}=\epsilon_2$, $\lambda_{2F}=\epsilon_3$, $\lambda_{2B}=\epsilon_4$. The case $(z_1,z_2,z_3,z_4)\to(z_1,z_1,z_2,z_2)$ corresponds to standard perturbation and we find

$$\langle \mathrm{str}\, AP(x)BP(y)\rangle = \frac{1}{4}\sum_{i,j=1,2}\Pi\left(x-y,\frac{\epsilon_i+\epsilon_j}{2}\right)(\mathrm{str}\, A\Lambda_i \mathrm{str}\, B\Lambda_j - \mathrm{str}\, A\Lambda_i\Sigma_z \mathrm{str}\, B\Lambda_j\Sigma_z + \mathrm{str}\, A\Lambda_i\Sigma_x \mathrm{str}\, B\Lambda_j\Sigma_x - \mathrm{str}\, A\Lambda_i\Sigma_y \mathrm{str}\, B\Lambda_j\Sigma_y),$$

$$\langle \mathrm{str}\, AP(x)\mathrm{str}\, BP(y)\rangle = \frac{1}{4}\sum_{i,j=1,2}\Pi\left(x-y,\frac{\epsilon_i+\epsilon_j}{2}\right)\mathrm{str}(A\Lambda_i B\Lambda_j - A\Lambda_i\Sigma_z B\Lambda_j\Sigma_z + A\Lambda_i\Sigma_x B\Lambda_j\Sigma_x - A\Lambda_i\Sigma_y B\Lambda_j\Sigma_y). \tag{A14}$$

APPENDIX B: CALCULATION OF THE TWO-LEVEL CORRELATION FUNCTION

1. Zero-mode integration

In this section we derive Eq. (95) by integrating the zero-mode variables of the nonperturbative parametrization (31). As before, the parametrization is slightly modified as

$$Q(x) = U_{\mathrm{u}}T_{\mathrm{ch}}T_{\mathrm{u}}^{(0)}\tilde{Q}(x)\overline{T}_{\mathrm{u}}^{(0)}\overline{T}_{\mathrm{ch}}\overline{U}_{\mathrm{u}}, \tag{B1}$$

to eliminate the Grassmann variables of the unitary part in F_2. For the preexponential term, dependence of the Grassmann variables on U_{u} is explicitly written as

$$\mathrm{str}\, k\Lambda_1\Sigma_z Q(x)\mathrm{str}\, k\Lambda_2\Sigma_z Q(x) \to \mathrm{str}\, k\Lambda_1\Sigma_z T_{\mathrm{ch}}\tilde{Q}(x)\overline{T}_{\mathrm{ch}}\mathrm{str}\, k\Lambda_2\Sigma_z T_{\mathrm{ch}}\tilde{Q}(x)\overline{T}_{\mathrm{ch}} - 4\xi\xi^*\,\eta\eta^*\,\mathrm{str}\,\Lambda_1\Sigma_z T_{\mathrm{ch}}T_{\mathrm{u}}^{(0)}\tilde{Q}(x)\overline{T}_{\mathrm{u}}^{(0)}\overline{T}_{\mathrm{ch}} \times \mathrm{str}\,\Lambda_2\Sigma_z T_{\mathrm{ch}}T_{\mathrm{u}}^{(0)}\tilde{Q}(x)\overline{T}_{\mathrm{u}}^{(0)}\overline{T}_{\mathrm{ch}}. \tag{B2}$$

The neglected terms do not contribute to integration of the Grassmann variables. The first term does not include the Grassmann variables ξ and η. We can set $T_{\mathrm{u}}^{(0)}=1$ and have

$$W_1(\epsilon_1,\epsilon_2) = \frac{1}{16\Delta^2V^2}\int \mathcal{D}Q\left[\int_x \mathrm{str}\, k\Lambda_1\Sigma_z Q(x)\right] \times\left[\int_y \mathrm{str}\, k\Lambda_2\Sigma_z Q(y)\right]e^{-F_2[Q]}, \tag{B3}$$

where $Q=T_{\mathrm{ch}}\tilde{Q}\overline{T}_{\mathrm{ch}}$. It still includes the zero-mode variables of the chiral part T_{ch}. Since the chiral part parametrization is the same as that of the DOS, the calculation can be done as in Sec. III B. As a result $W_1(\epsilon_1,\epsilon_2)$ in Eq. (95) is obtained. It includes a perturbative part and connected and disconnected parts.

Next we consider the second contribution which includes only the connected part. It is obtained by integrations of ξ and η as

$$W_2(\epsilon_1,\epsilon_2) = \frac{1}{\Delta^2}\int ds_1 ds_2\frac{d\varphi_B}{2\pi}\frac{d\varphi_F}{2\pi}\frac{4s_1s_2}{(s_1^2-s_2^2)^2}I(z_{1,2},s_{1,2},\varphi_{B,F}),$$

$$I(z_{1,2},s_{1,2},\varphi_{B,F}) = -\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\int \mathcal{D}\tilde{Q}\mathcal{D}Q_{\mathrm{ch}}e^{-F[Q]}, \tag{B4}$$

where $Q(x)=T_{\mathrm{ch}}T_{\mathrm{u}}^{(0)}\tilde{Q}(x)\overline{T}_{\mathrm{u}}^{(0)}\overline{T}_{\mathrm{ch}}$, $Q_{\mathrm{ch}}=T_{\mathrm{ch}}\Sigma_z\overline{T}_{\mathrm{ch}}$, and $z_{1,2}=\pi\epsilon_{1,2}/\Delta$. We examine $\mathrm{str}\,\hat{\epsilon}\Sigma_z Q(x)$ to integrate out variables in Q_{ch}. The expression is simplified if we apply the saddle-point approximation we use in the following. At the saddle point we have $\sin\hat{\theta}=0$ and $\sin\hat{\Omega}=0$. This approximation leads to the reduction

$$\mathrm{str}\,\hat{\epsilon}\Sigma_z Q(x) \to \epsilon_1\mathrm{str}\frac{\cos\hat{\Omega}+\Lambda}{2}[\cos\hat{\theta}_1+(\cosh\theta_{1B}-\cos\theta_{1F})\rho_1\sigma_1]\Sigma_z\tilde{Q}(x) + \epsilon_2\mathrm{str}\frac{\cos\hat{\Omega}-\Lambda}{2}[\cos\hat{\theta}_2+(\cosh\theta_{2B}-\cos\theta_{2F})\rho_2\sigma_2]\Sigma_z\tilde{Q}(x). \tag{B5}$$

Again we stress that this approximation is justified at second order in $1/g$. Substituting this expression, we have

$$I(z,s,\varphi) = -\int \mathcal{D}\tilde{Q}e^{-F_{\mathrm{kin}}}I_1(z_1,s)I_2(z_2,s), \tag{B6}$$

$$I_{i=1,2}(z,s)=\frac{\partial}{\partial z}\int\mathcal{D}Q_{\rm ch}\left[1-\frac{iz}{2V}\int_x \mathrm{str}\frac{\cos\hat{\Omega}\pm\Lambda}{2}(\cosh\theta_{iB}-\cos\theta_{iF})\rho_i\sigma_i\Sigma_z\tilde{Q}(x)\right]\exp\left[-\frac{iz}{2V}\times\int_x\mathrm{str}\frac{\cos\hat{\Omega}\pm\Lambda}{2}\cos\hat{\theta}_i\Sigma_z\tilde{Q}(x)\right], \qquad (B7)$$

where $F_{\rm kin}$ is the kinetic part in F_2. The variable φ is not included in the integrand in our approximation. Integrations of the remaining zero-mode variables are carried out and we find

$$\begin{aligned}I_i(z,s)&=i(s_1-s_2+C_i)\Bigg\{e^{iz(s_1-s_2)+izC_i}+\frac{1}{4\pi}\int d\theta_{iB}d\theta_{iF}\frac{\lambda_{iB}\lambda_{iF}-1}{\lambda_{iB}-\lambda_{iF}}\left[1+iz\left(s_1\lambda_{iB}-s_2\lambda_{iF}+\frac{1}{2}(C_i-D_i)\lambda_{iB}+\frac{1}{2}(C_i+D_i)\lambda_{iF}\right)\right]\exp\left[iz\left(s_1\lambda_{iB}-s_2\lambda_{iF}+\frac{1}{2}(C_i-D_i)\lambda_{iB}+\frac{1}{2}(C_i+D_i)\lambda_{iF}\right)\right]\Bigg\}\\&=\frac{i\pi z}{2}(s_1-s_2+C_i)(s_1+s_2-D_i)J_0\left(zs_1+z\frac{C_i-D_i}{2}\right)J_0\left(zs_2-z\frac{C_i+D_i}{2}\right),\end{aligned} \qquad (B8)$$

where $\lambda_{iF}=\cos\theta_{iF}$ and $\lambda_{iB}=\cosh\theta_{iB}$. This result yields W_2 in Eq. (95).

2. AA's domain

We consider Eq. (95) in the AA domain $1\ll z_{1,2}$ using the asymptotic form of the Bessel function (60). For W_1,

$$\begin{aligned}W_1(\epsilon_1,\epsilon_2)&\sim\frac{1}{\Delta^2}\langle[f_1(z_1)e^{iz_1(A_{B1}+A_{F1})}+g_1(z_1)e^{2iz_1+iz_1(A_{B1}-A_{F1})}]\times[f_2(z_2)e^{iz_2(A_{B2}+A_{F2})}+g_2(z_2)e^{2iz_2+iz_2(A_{B2}-A_{F2})}]\rangle_{\rm kin}\\&=\frac{1}{\Delta^2}\int\mathcal{D}\tilde{Q}[e^{-F(z_1,z_1,z_2,z_2)}f_1(z_1)f_2(z_2)+e^{2iz_1-F(-z_1,z_1,z_2,z_2)}g_1(z_1)g_2(z_2)+e^{2iz_2-F(z_1,z_1,-z_2,z_2)}f_1(z_1)g_2(z_2)+e^{2i(z_1+z_2)-F(-z_1,z_1,-z_2,z_2)}g_1(z_1)g_2(z_2)],\end{aligned} \qquad (B9)$$

where

$$f_i(z)=1+\frac{1}{8z^2}+\frac{1}{2}(A_{Bi}-A_{Fi})+\cdots,$$

$$g_i(z)=-\frac{1}{2z}+\frac{i}{8z^2}+\cdots, \qquad (B10)$$

and F is the supersymmetry breaking functional

$$F(z_1,z_2,z_3,z_4)=\frac{\pi D}{4\Delta V}\int\mathrm{str}(\nabla\tilde{Q})^2+\frac{i}{2V}\int\mathrm{str}(z_1k_F+z_2k_B)\Lambda_1\Sigma_z(\tilde{Q}-\Sigma_z)+\frac{i}{2V}\int\mathrm{str}(z_3k_F+z_4k_B)\Lambda_2\Sigma_z(\tilde{Q}-\Sigma_z). \qquad (B11)$$

The condition $z_1=z_2$ and $z_3=z_4$ recovers supersymmetry. As before we expand the nonzero modes $\tilde{Q}$ in terms of the P matrix and use the contraction rules derived in Appendix A. The first term in Eq. (B9) does not break supersymmetry $[F(z_1,z_1,z_2,z_2)=F_2]$ and is nothing but the purely perturbative contribution. Its connected part gives the first term in Eq. (98). The second (third) term in Eq. (B9) gives the second (third) term in Eq. (98). For the leading order contribution, we use

$$\int\mathcal{D}\tilde{Q}e^{-F(-z_1,z_1,z_2,z_2)}\sim\mathcal{D}_1,$$

$$\langle\mathrm{str}\,k\Lambda_2P^2\rangle_{F^{(0)}(-z_1,z_1,z_2,z_2)}=-2\sum_{q\neq0}(\Pi_+-\Pi_-^*),$$

$$\int\mathcal{D}\tilde{Q}e^{-F(z_1,z_1,-z_2,z_2)}\sim\mathcal{D}_2,$$

$$\langle\mathrm{str}\,k\Lambda_1P^2\rangle_{F^{(0)}(z_1,z_1,-z_2,z_2)}=-2\sum_{q\neq0}(\Pi_+-\Pi_-). \qquad (B12)$$

For the last term in Eq. (B9), we have

$$\int\mathcal{D}\tilde{Q}e^{-F(-z_1,z_1,-z_2,z_2)}\sim\mathcal{D}_1\mathcal{D}_1\mathcal{D}_+^2\mathcal{D}_-^{-2}. \qquad (B13)$$

The disconnected part is included in this contribution and is subtracted to give the fourth and fifth terms in Eq. (98).

The purely connected part W_2 is calculated using the asymptotic form of the Bessel function. We obtain

$$\begin{aligned}W_2(\epsilon_1,\epsilon_2)\sim\frac{1}{4\Delta^2}\int_1^\infty ds_1\int_0^1 ds_2\langle&\{e^{iz_1[s_1+(s_1A_-+A_-^{(\Lambda)})/2]-i\pi/4}+(*)\}\\&\times\{e^{iz_1[s_2-(s_2A_++A_+^{(\Lambda)})/2]-i\pi/4}+(*)\}\\&\times\{e^{iz_2[s_1+(s_1A_--A_-^{(\Lambda)})/2]-i\pi/4}+(*)\}\\&\times\{e^{iz_2[s_2-(s_2A_+-A_+^{(\Lambda)})/2]-i\pi/4}+(*)\}\rangle_{\rm kin},\end{aligned} \qquad (B14)$$

where $(*)$ denotes the complex conjugate of the preceding term. Integrations of $s_{1,2}$ are evaluated to find the asymptotic form

$$W_2(\epsilon_1,\epsilon_2) \sim -\frac{1}{4\Delta^2}\int \mathcal{D}\tilde{Q}\Bigg\{\frac{1}{(z_1-z_2)^2}\big[e^{-F(z_1,z_1,-z_2,-z_2)} - e^{2i(z_1-z_2)-F(-z_1,z_1,z_2,-z_2)} + (z_{1,2}\to -z_{1,2})\big] + \frac{1}{(z_1+z_2)^2}\big[e^{-F(z_1,z_1,z_2,z_2)} + e^{2i(z_1+z_2)-F(-z_1,z_1,-z_2,z_2)} + (z_{1,2}\to -z_{1,2})\big] + \frac{i}{z_1^2-z_2^2}\big[e^{2iz_1-F(-z_1,z_1,z_2,z_2)} + e^{2iz_1-F(-z_1,z_1,-z_2,-z_2)} + e^{-2iz_2-F(z_1,z_1,z_2,-z_2)} + e^{-2iz_2-F(-z_1,-z_1,z_2,-z_2)} - (z_{1,2}\to -z_{1,2})\big]\Bigg\}. \tag{B15}$$

Finally, keeping second order in P for the functional F and using the formula (A12), we obtain the result (99).

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